ECE1240

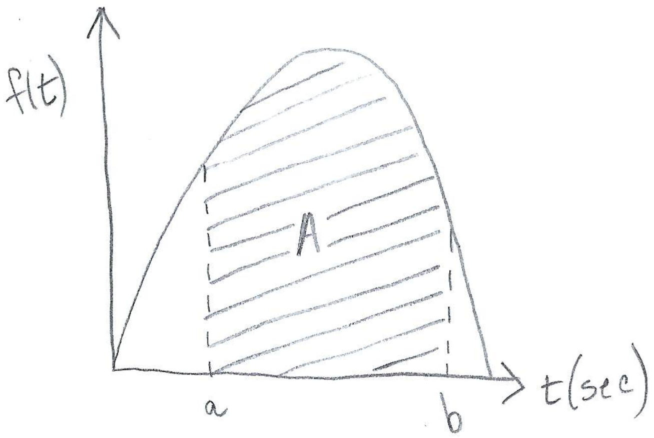
Integrals in Electric Circuits

An *integral* is an inverse function of a derivative.

$\Rightarrow F(t) = \int f(t)\,dt = \int \frac{d}{dt} F(t)\,dt = F(t)$

$\Rightarrow F(t)$ is also called the antiderivative of $f(t)$

Consider that for a function, $f(t)$, the definite integral $\int_a^b f(t)\,dt$ is the area under the function $f(t)$ between point a & point b



Area $A = \int_a^b f(t)\,dt = F(t)\Big|_a^b = F(b) - F(a)$

Indefinite integrals have no limits (a&b)

$\Rightarrow \int 3t^2\,dt$ is indefinite

The three typical categories of functions, $f(t)$, that may require differentiation in electrical engineering problems are the same as those that may require integration (taking an integral):

1) Polynomials, such as quadratic equations $f(t) = 4t^2 + 6t + 5$ as an example
2) Exponentials, such as $f(t) = 4e^{-t}$
3) sin and cos functions, such as $f(t) = \cos(2\pi t)$

Steps for taking the integral of these are as follows:

1) Consider $f(t) = 4t^2 + 6t + 5$
   i) Integrate each term separatly
   $$\Longrightarrow \int f(t) = \int 4t^2\,dt + \int 6t\,dt + \int 5\,dt$$
   ii) Separate coefficients from variable terms
   $$\Longrightarrow \int f(t) = 4 \cdot \int t^2\,dt + 6 \cdot \int t\,dt + 5 \cdot \int dt$$
   iii) Use the power rule for integrals (reverse of derivatives) to take the integral of each term. Increase the power by 1, and divide by the new power. Coefficients remain out in front.

$$\Rightarrow 4\cdot\int t^2\,dt = 4\cdot\frac{t^{2+1}}{2+1} = \frac{4t^3}{3}$$

power+1

new power

Note: This power rule holds for all $f(t)$ that are powers of t, except when the original power is -1.

iv) Integral of term with only "dt" will be "t"

$$\Rightarrow 5\cdot\int dt = 5\cdot t = 5t$$

v) Indefinite integrals must include a constant of integration, C as the inverse for the derivative of a constant

$$\Rightarrow \int(4t^2+6t+5)\,dt = 4\cdot\int t^2\,dt + 6\cdot\int t\,dt + 5\cdot\int dt$$

$$\Rightarrow \int(4t^2+6t+5)\,dt = 4\cdot\frac{t^3}{3} + 6\cdot\frac{t^2}{2} + 5\cdot\frac{t^1}{1} + C$$

power +1

new power

$$= \frac{4t^3}{3} + 3t^2 + 5t + C$$

2) Consider $f(t) = 4e^{-t}$

i) Separate coefficients from exponential terms

$$\Rightarrow f(t) = 4\cdot e^{-t}$$

ii) The integral of an exponential is the exponential term divided by the coefficient of the variable.

$\Rightarrow \int 4e^{-t}\,dt = 4\cdot\int e^{-t}dt = 4\cdot\frac{e^{-t}}{-1}$

iii) Include a constant of integration, C, as the inverse for the derivative of a constant

$\Rightarrow \int 4e^{-t}dt = 4\cdot\int e^{-t}dt = 4\cdot\frac{e^{-t}}{-1} + C$

$= -4e^{-t} + C$

3) Consider $f(t) = 2\cos(2\pi t)$

i) Separate coefficients from trig terms

$\Rightarrow f(t) = 2\cdot\cos(2\pi t)$

ii) Separate coefficient(s) of the variable within the trig function from the variable

$\Rightarrow f(t) = 2\cdot\cos[(2\pi)\cdot t]$

iii) Use the following to take the integral of the trig term:

if $f(t) = \sin(at) \Rightarrow \int \sin(at) = -\frac{1}{a}\cos(at)$

if $f(t) = \cos(at) \Rightarrow \int \cos(at) = \frac{1}{a}\sin(at)$

$\Rightarrow \int 2\cos(2\pi t)\,dt = 2\cdot\int \cos(2\pi t)\,dt$

$= (\cancel{2})\left(\frac{1}{\cancel{2}\pi}\right)\sin(2\pi t) = \frac{\sin(2\pi t)}{\pi}$

iv) Include a constant of integration, C, as the inverse for the derivative of a constant.

$$\Longrightarrow \int 2\cos(2\pi t)\,dt = 2\cdot\int \cos(2\pi t)\,dt$$

$$\Longrightarrow 2\cdot\int \cos(2\pi t)\,dt = (\cancel{2})\left(\frac{1}{\cancel{2}\pi}\right)\sin(2\pi t)$$

$$= \frac{\sin(2\pi t)}{\pi} + C$$

Taking the derivative of a final answer can serve as a check to be sure the integral is correct

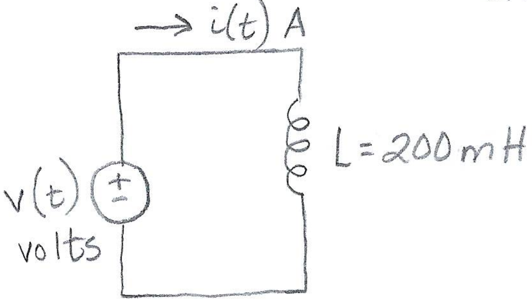
$$\Longrightarrow \frac{d}{dt}\left[\left(\frac{\sin(2\pi t)}{\pi}\right) + C\right] = \left(\frac{1}{\pi}\right)\frac{d}{dt}\left[\left(\sin(2\pi\cdot t)\right) + C\right]$$

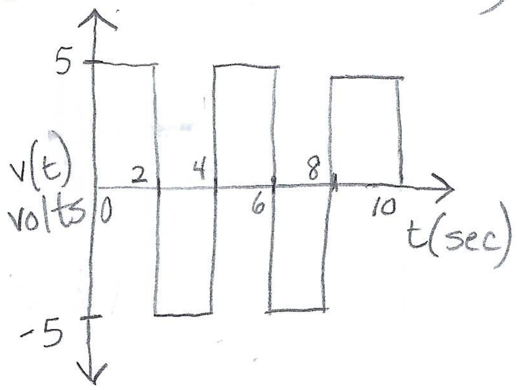
$$= \left(\frac{1}{\pi}\right)(2\pi)\cos(2\pi t) + \frac{d}{dt}(C)$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi}}\cos(2\pi t) + 0$$

$$= 2\cos(2\pi t) = f(t) \checkmark$$

Since integrals and derivatives are inverse functions, math based relationships containing integrals also exist for electric circuits, just the same as derivatives. Consider the circuit below where current and voltage are functions of time, t, and v(t) is defined by the graph: (See Class Module 4 and Chapter 5-3 of textbook)





Using the relationship:

$$v(t) = L\frac{di}{dt}$$

$i(t)$ can be found as follows:

$$\frac{di}{dt} = \left(\frac{1}{L}\right)v(t)$$

$$\Rightarrow \int_0^t \frac{di}{dt}\,dt = i(t)\Big|_0^t = i(t) - i(0) = \int_0^t \frac{1}{L}v(t)\,dt$$

From the graph, $v(t) = 5$ volts for $0 \le t(\text{sec}) \le 2$

$$\Rightarrow i(2) - i(0) = \int_0^2 \left(\frac{1}{200\,mH}\right)(5\,dt) = \left(\frac{1}{200\,mH}\right)(5t)\Big|_0^2 = \frac{10}{.2} = 50\,A$$

Assuming $i(0) = 0$

$$\Rightarrow i(2) = 50A + i(0) = 50A + 0A = 50A$$

From the graph, $v(t) = -5$ volts for $2 \le t(\text{sec}) \le 4$

$$\Rightarrow i(4) - i(2) = \int_2^4 \left(\frac{1}{200\,mH}\right)(-5\,dt) = \left(\frac{1}{200\,mH}\right)(-5t)\Big|_2^4 = -100A + 50A$$

$$\Rightarrow i(4) = i(2) - 50A = 50A - 50A = 0$$

From the graph, $v(t) = 5$ volts for $4 \le t(\text{sec}) \le 6$

$$\Rightarrow i(6) - i(4) = \int_4^6 \left(\frac{1}{200\,mH}\right)(5\,dt) = \left(\frac{1}{200\,mH}\right)(5t)\Big|_4^6 = 150A - 100A$$

$$\Rightarrow i(6) = i(4) + 50A = 0A + 50A = 50\,A$$

From the graph, $v(t) = -5$ volts for $6 \le t(\text{sec}) \le 8$

$$\Rightarrow i(8) - i(6) = \int_6^8 \left(\frac{1}{200\,mH}\right)(-5\,dt) = \left(\frac{1}{200\,mH}\right)(5t)\Big|_6^8 = -200A + 150A$$

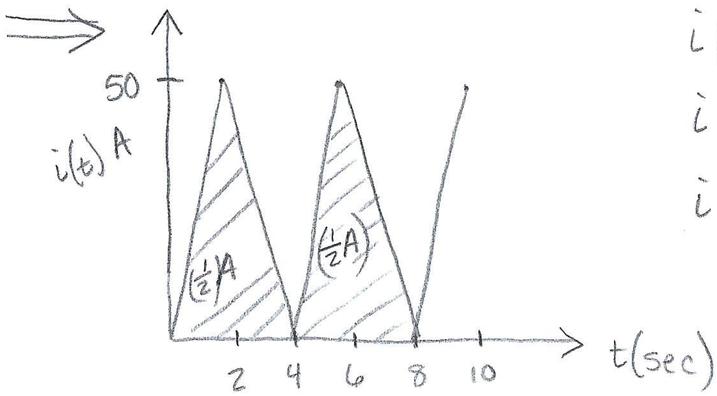
$$\Rightarrow i(8) = i(6) - 50\,A = 50\,A - 50\,A = 0\,A$$

From the graph, $v(t) = 5$ volts for $8 \le t(\text{sec}) \le 10$ and will replicate $v(t) = 5$ volts for $4 \le t(\text{sec}) \le 6$

$$\Rightarrow i(10) = i(8) + 50\,A = 50\,A$$

From this we can graph $i(t)$

$\Rightarrow$



$i(0) = 0\,A$, $i(2) = 50\,A$
$i(4) = 0\,A$, $i(6) = 50\,A$
$i(8) = 0\,A$, $i(10) = 50\,A$

To find the total current through the circuit for $0 \le t(\text{sec}) \le 8\,\text{sec}$, determine area A for the interval $0 \le t \le 8$

$$\Rightarrow \left(\tfrac{1}{2}\right)(\text{base})(\text{height}) \text{ for } 0 \le t \le 4 = \left(\tfrac{1}{\cancel{2}}\right)(\cancel{4}^{\,2})(50) = 100\,A$$

$$\Rightarrow i(t) \text{ for } 0 \le t \le 8 = (2)(100) = 200\,A$$

Integral Practice Problems and Solutions

Find $q(t)$ flowing through a device if $q(t) = \int i(t)\,dt + q(0)$ and the current $i(t)$ is:

a) $i(t) = 20\cos(10t + \pi/6)$ A, $q(0) = 0$

b) $i(t) = (3t^2 + 4t + 5)$ A, $q(0) = 1$ C

c) $i(t) = (10e^{-30t})$ A, $q(0) = 2$C

Solutions:

a) $q(t) = \int i(t)\,dt + q(0)$

$$= \int 20\cos(10t + \pi/6)\,dt + q(0)$$

Let $x = 10t + \pi/6 \Rightarrow \frac{dx}{dt} = \frac{d}{dt}(10t + \pi/6) = 10$

$$\Rightarrow dt = \frac{1}{10}dx$$

$$\int 20\cos[(x)(\tfrac{1}{10})]dx + q(0) = \int \frac{20}{10}\cos x\,dx + q(0)$$

$$\int 2\cos(x)\,dx + q(0) = 2\int \cos x\,dx + q(0)$$

$$2\int \cos(x)\,dx + q(0) = 2\sin(x) + 0$$

Substitute $x = 10t + \pi/6$

$$\Rightarrow 2\sin(x) = 2\sin(10t + \pi/6)$$

$$\Rightarrow q(t) = 2\sin(10t + \pi/6)\ \text{C}$$

Solutions (cont):

b) $q(t) = \int i(t)\,dt + q(0)$

$= \int (3t^2 + 4t + 5)\,dt + 1$

$= 3\int t^2\,dt + 4\int t\,dt + 5\int dt + 1$

$\Rightarrow (3)\dfrac{t^{2+1}}{2+1} + (4)\dfrac{t^{1+1}}{1+1} + (5)(t) + 1$

$\Rightarrow q(t) = \dfrac{3t^3}{3} + \dfrac{4t^2}{2} + 5t + 1$

$\Rightarrow q(t) = (t^3 + 2t^2 + 5t + 1)\,C$

c) $q(t) = \int i(t)\,dt + q(0)$

$= \int (10e^{-30t})\,dt + 2$

Let $x = -30t \Rightarrow \dfrac{dx}{dt} = \dfrac{d}{dt}(-30t) = -30$

$\Rightarrow dt = -\frac{1}{30}\,dx$

$\int 10e^x\left(-\frac{1}{30}\,dx\right) + q(0) = \int \left(-\frac{10}{30}\right)e^x\,dx + q(0)$

$\int \left(-\frac{1}{3}\right)e^x\,dx + q(0) = -\frac{1}{3}\int e^x\,dx + q(0)$

$-\frac{1}{3}\int e^x\,dx + q(0) = -\frac{1}{3}e^x + 2$

Substitute $x = -30t$

$\Rightarrow -\frac{1}{3}e^x + 2 = -\frac{1}{3}e^{-30t} + 2$

$\Rightarrow q(t) = \left(-\frac{1}{3}e^{-30t} + 2\right)C$