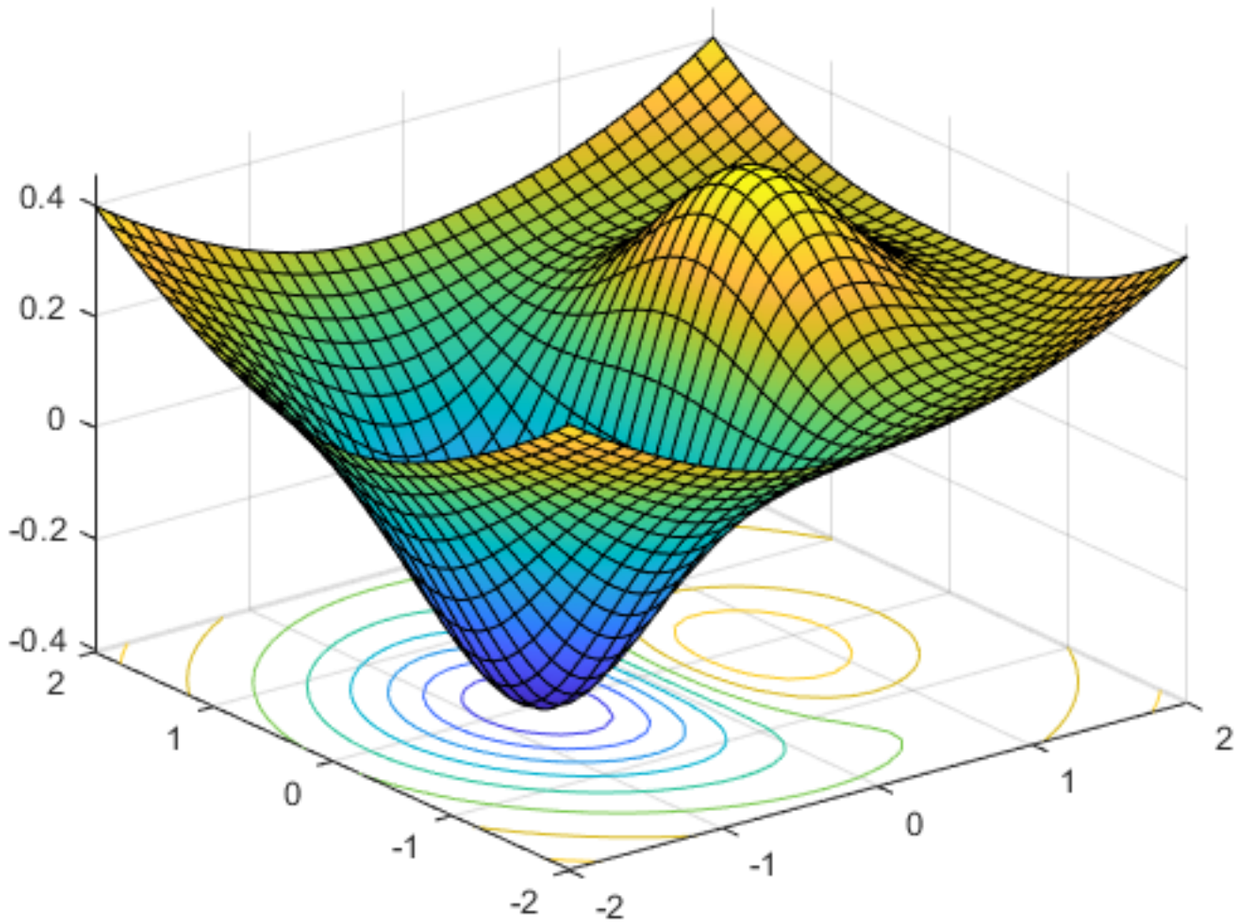
Advanced Algorithms

Lecture 21: Optimization and linear programming

Announcements

- HW 5 out — due Wednesday Nov 20
- Next Tuesday (Nov 12): Guest lecture by Suresh Venkat on **Algorithmic Fairness**

Optimization



Goal: maximize or minimize a function over a “domain”

Problems –> optimization

Set cover

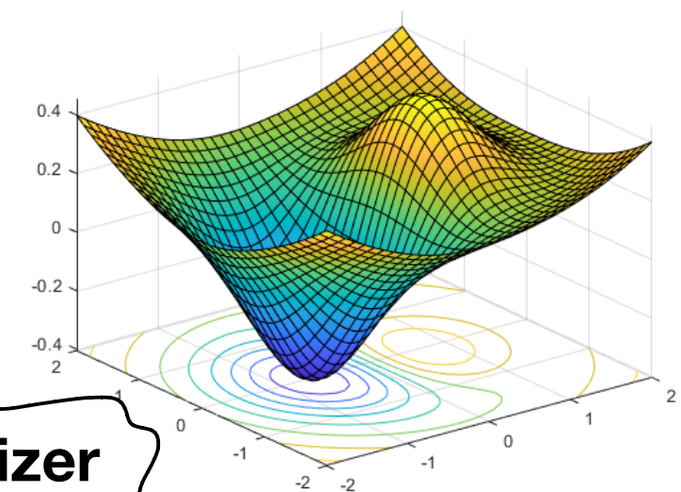
Instance of problem

Optimization formulation

variables x_1, x_2, …
constraints
objective

Optimizer

Solver.



opt solution:
$x_1 = 0$, $x_2 = 1$, …

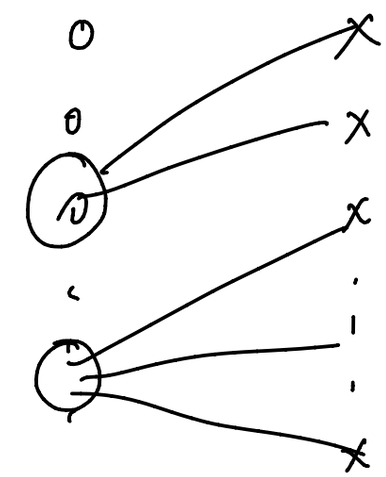
Solution to problem instance

Important aspects

- How to choose variables? (must help us obtain solution to original)
- Optimizer knows nothing about original problem or instance (only knows constraints, variables, ...) — requires proof/argument
- Important to ensure that opt problem can be solved efficiently:
  - how many variables/constraints?
  - discrete/continuous domain
  - type of constraints & objective (linear, quadratic, etc.)

Examples from last class

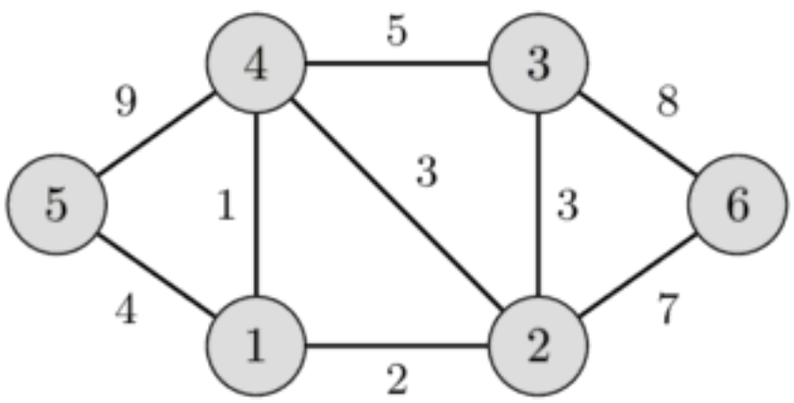
- Set Cover
- Matching — assigning gifts to children
- Minimum spanning tree



Spanning tree

Problem: let G = (V, E) be a (simple, undirected) graph with edge weights $\{w_e\}$ (>0). Pick a subset of the edges, such that (a) all vertices are "connected", (b) total weight of edges is minimized

- What are variables?
- Constraints?
- Objective?



Easy: x_e for each edge, minimize sum of $w_e x_e$

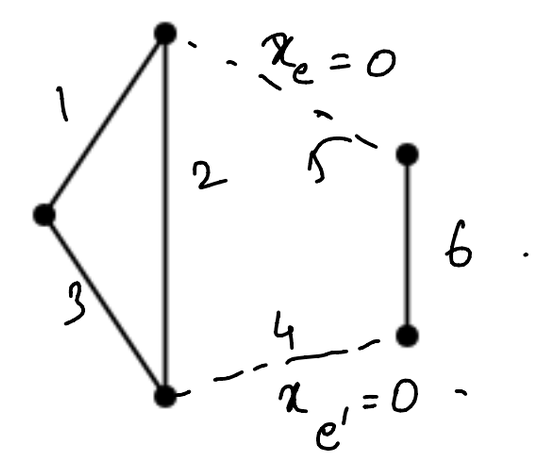
Constraints for MST

Variables: x_e : (indicates if e is picked or not) — binary.

Ensuring that every vertex is connected to every other …

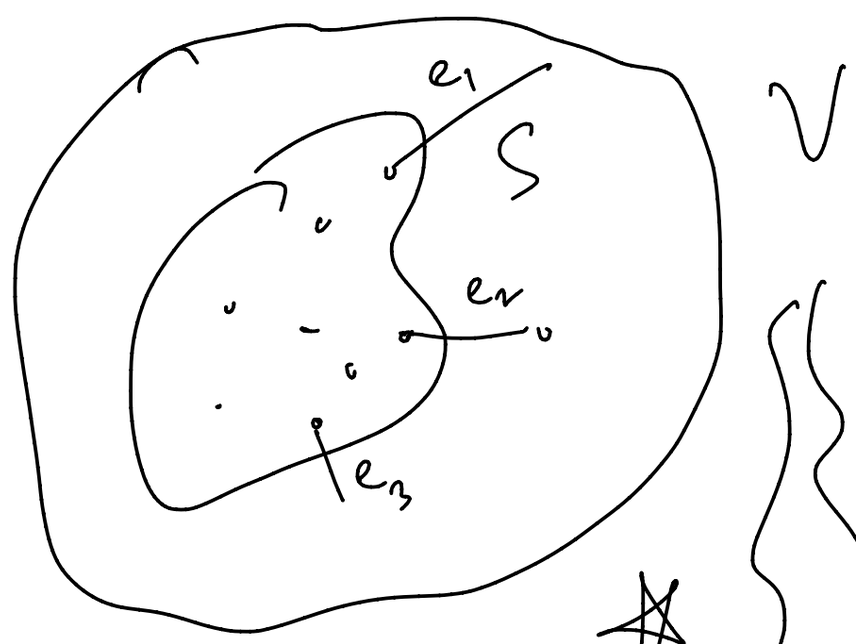
- Try 1: total number of edges = n-1 $\quad \sum_e x_e = n-1$
- Try 2: add constraint that one edge out of every vertex is chosen
- Try 3: in every cycle of size *r*, make sure at most *(r-1)* edges are chosen — exponentially many constraints
- Alternate: make sure at least one path from u — v is "present" for each pair uv (constraints aren't linear any more)

$$x_1 + x_2 + x_3 \le 2$$



"Standard" formulation for MST

- One constraint for every subset of vertices
- Idea: for any subset, at least one edge going "out of it" must be in MST



$$\begin{cases} \forall T: \sum_{e \in \text{Edges going out of } T} x_e \geq 1 \\ \text{each } x_e \in \{0,1\} \\ \min \sum_e w_e x_e \end{cases}$$

$2^{|V|-1}$ constraints.

"Validity" of formulation

- **Must prove:** any feasible solution to set of constraints yields a *valid* solution to original problem .

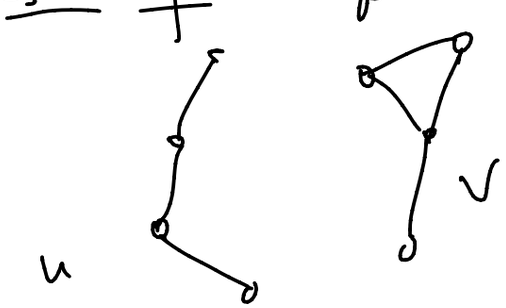
Consider any feasible solution to $\star$. $\longrightarrow x$ ('bunch of x_e vals).

Claim: if S denotes the set of edges for which $x_e = 1$,

then S includes a spanning tree.

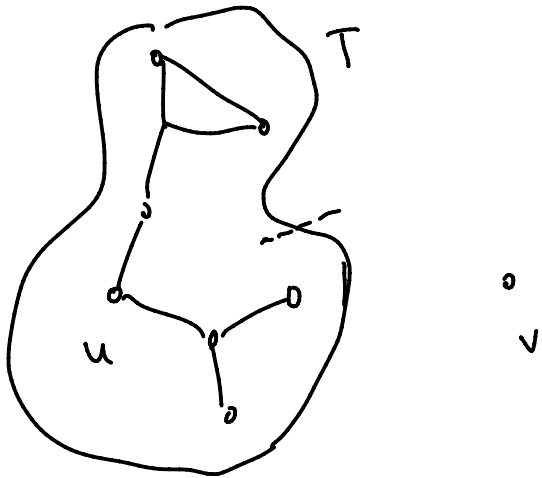
$\Updownarrow$

the graph with just the edges in S contains a path from between any two vertices u, v.



We <u>will argue</u>: if there is no path from $u \to v$ then there exists a constraint that is violated.

(contradicts fact that x was a feasible solution).



- Define T = set of vertices reachable from u using only the edges S.

- <u>Obs</u>: T does not contain v.

- Obs: none of the edges out of T could have $x_e = 1$.

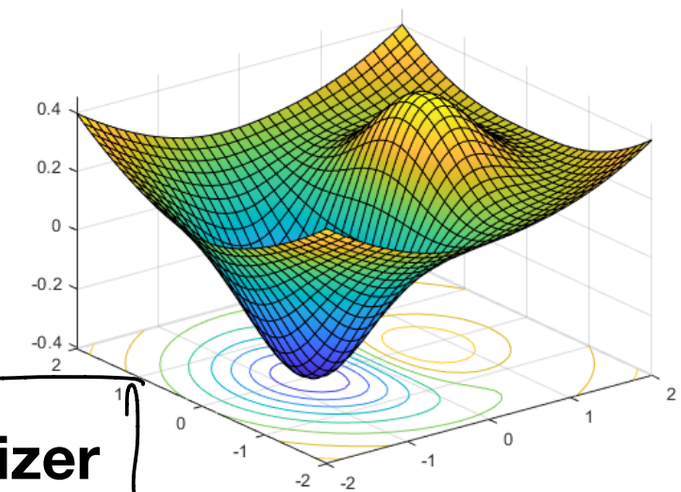
- To prove validity of formulation, we must also ensure that for all feasible solutions to original problem, we have a solution to the opt problem. [not "over constraining"]

Recap

Instance of problem

Optimization formulation

variables x_1, x_2, …
constraints
objective

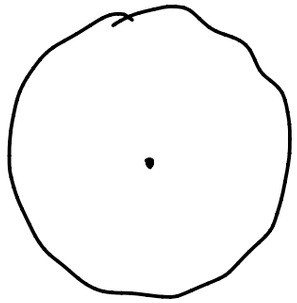


Optimizer

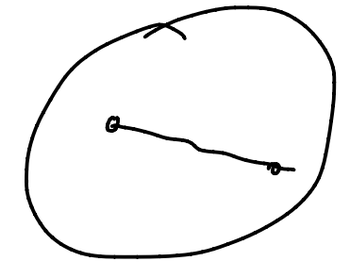
opt solution:
$x_1 = 0$, $x_2 = 1$, …

Solution to problem instance

$\sum a_i x_i \le b_i$



Efficiency of optimizers



What optimization problems can be solved efficiently?

provably efficient.

- Linear programs — real valued variables, linear constraints, linear objective

  (time poly(#variables, #constraints).

- "Convex" programs — real valued variables, convex constraints, *minimize* convex objective

  feasible set is convex.

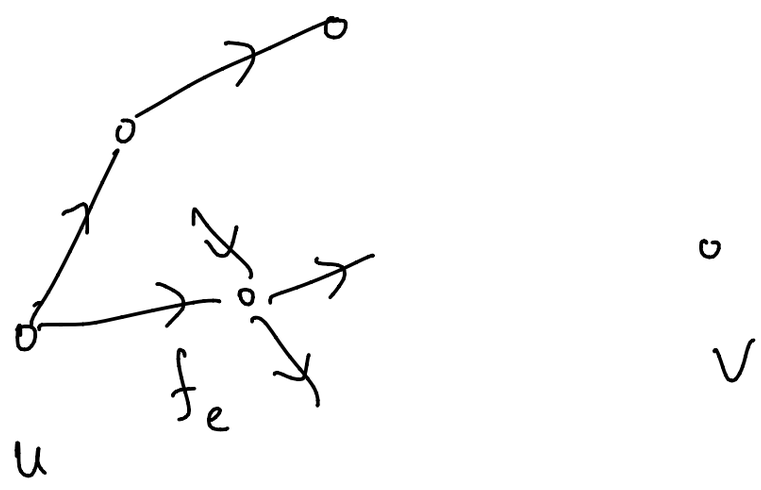
- Good heuristics for "integer linear programs" (esp binary valued), "quadratic programming", ...

  QCQP.

Most optimizers are iterative algorithms — constantly improve solution

Linear programming

- Clean geometric interpretation
- Many "operations research" problems (e.g. "maximum flow")



flow: $f_e \geq 0$

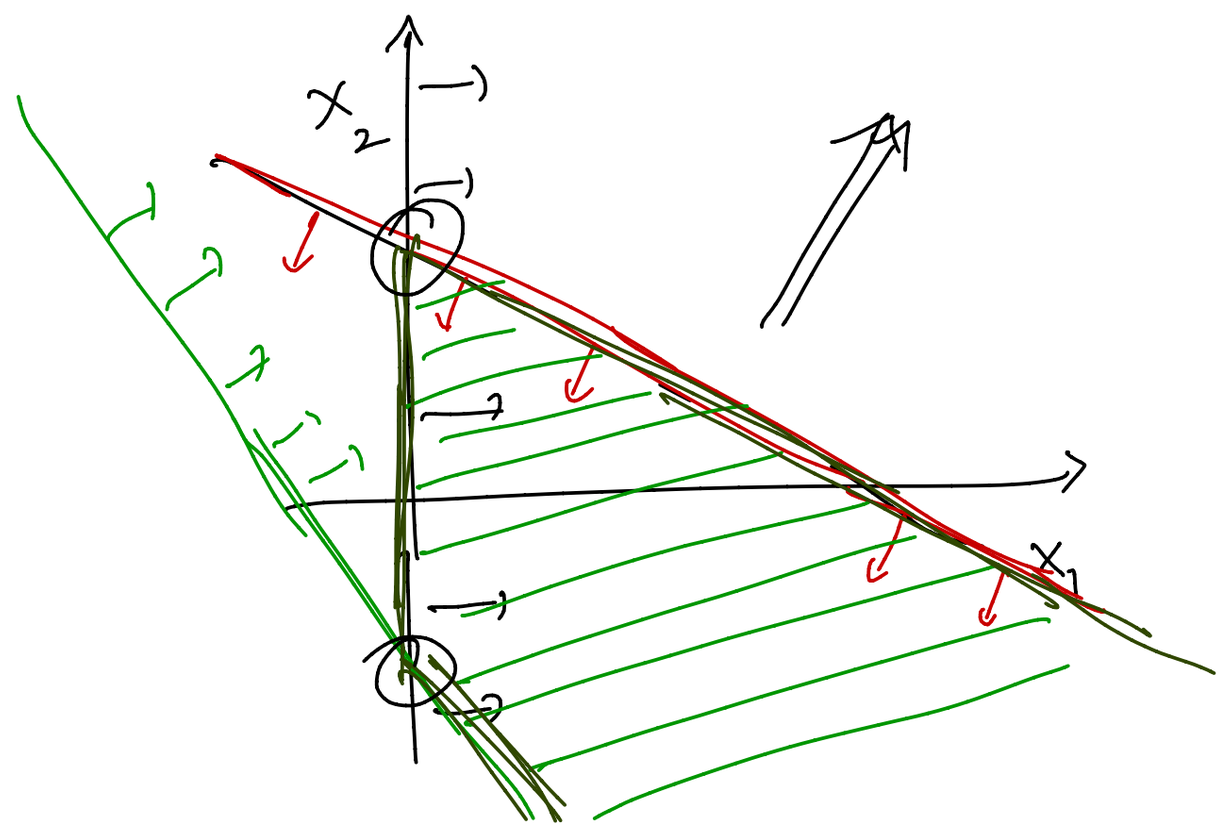
- value on every edge
- total in-flow $\equiv$ total out-flow. (for all vertices $\neq u, v$)
- $f_e \leq cap(e)$.

Geometry of LP

- Constraints half-spaces
- Feasible region "polytope"

minimize $x_1 + x_2$ subject to

$$\begin{cases} x_1 >= 0 \\ x_1 + 3x_2 <= 4 \\ 2x_1 + x_2 >= -1 \end{cases}$$



Simplex algorithm

- Local search on the "corners" of polytope
- Works well in practice → "Simplex".

[optimum value is either $\pm\infty$ or is attained at some "corner" of the polytope

The Great Mathematical Sputnik of 1979

Eugene L. Lawler

University of California, Berkeley

I

Under the frontpage headline, "A Soviet Discovery Rocks World of Mathematics," the New York *Times* of November 7, 1979 announced an event which its readers could easily believe had the importance of the launching of Sputnik. "A surprise discovery by an obscure Soviet mathematician," said the *Times*, "has rocked the world of mathematics and computer analysis ... Apart from its profound theoretical interest, the new discovery may be applicable in weather prediction, complicated industrial processes, petroleum refining, the scheduling of workers at large factories ..." Furthermore, confided the *Times*, "the theory of [secret] codes could eventually be affected by the Russian discovery, and this fact has obvious importance to intelligence agencies everywhere." One could almost hear alarm bells ringing in the offices of the CIA and NSA.

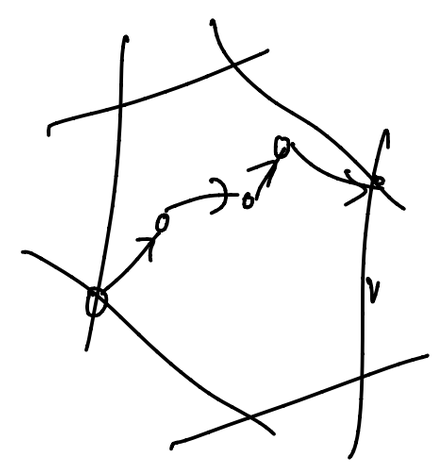
Simplex algorithm

- Local search on the "corners" of polytope
- Works well in practice
- Khachiyan's algorithm [1979]
- Karmarkar's algorithm [1984]

(Bell labs).

polynomial time.

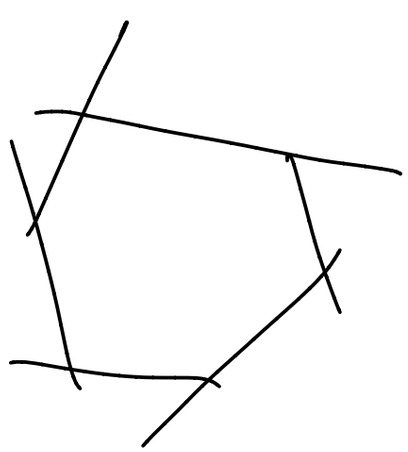
"Interior point methods".



What about combinatorial problems?

- All reductions we've seen involve "binary" variables
- "Relaxations" of constraints

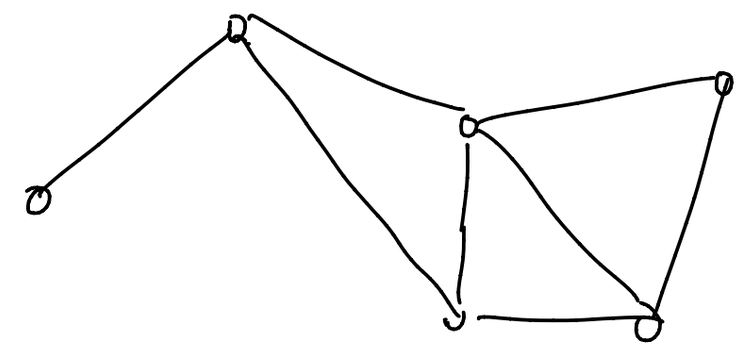
Can LPs be useful for solving combinatorial problems?



- For some problems, <u>all corners</u> are binary-valued.

Vertex cover relaxation

- Recap: HW 3



goal is to select a subset of vertices of min possible size, s.t. every edge is "covered" (i.e., one of its end pts is selected)

Variables:

for each vertex u, have variable $x_u \in \{0,1\}$ $\rightarrow 0 \le x_u \le 1$

Constraints:

for every edge $\{uv\}$, $x_u + x_v \geq 1$

Obj:

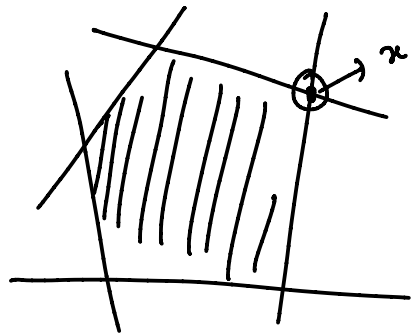
$$\min \sum_u x_u$$

Original opt formulation:

- objective
- constraints
- $x_u \in \{0,1\}$

LP "relaxation"

- obj..
- constraints $\boxed{x_i + x_j \geq 1}$
- $\boxed{0 \leq x_u \leq 1}$



Suppose for every corner point x, $x_u \in \{0,1\}$.

$\downarrow$

then we're in good shape!

“Cheating” solutions

Rounding, approximation

Set cover