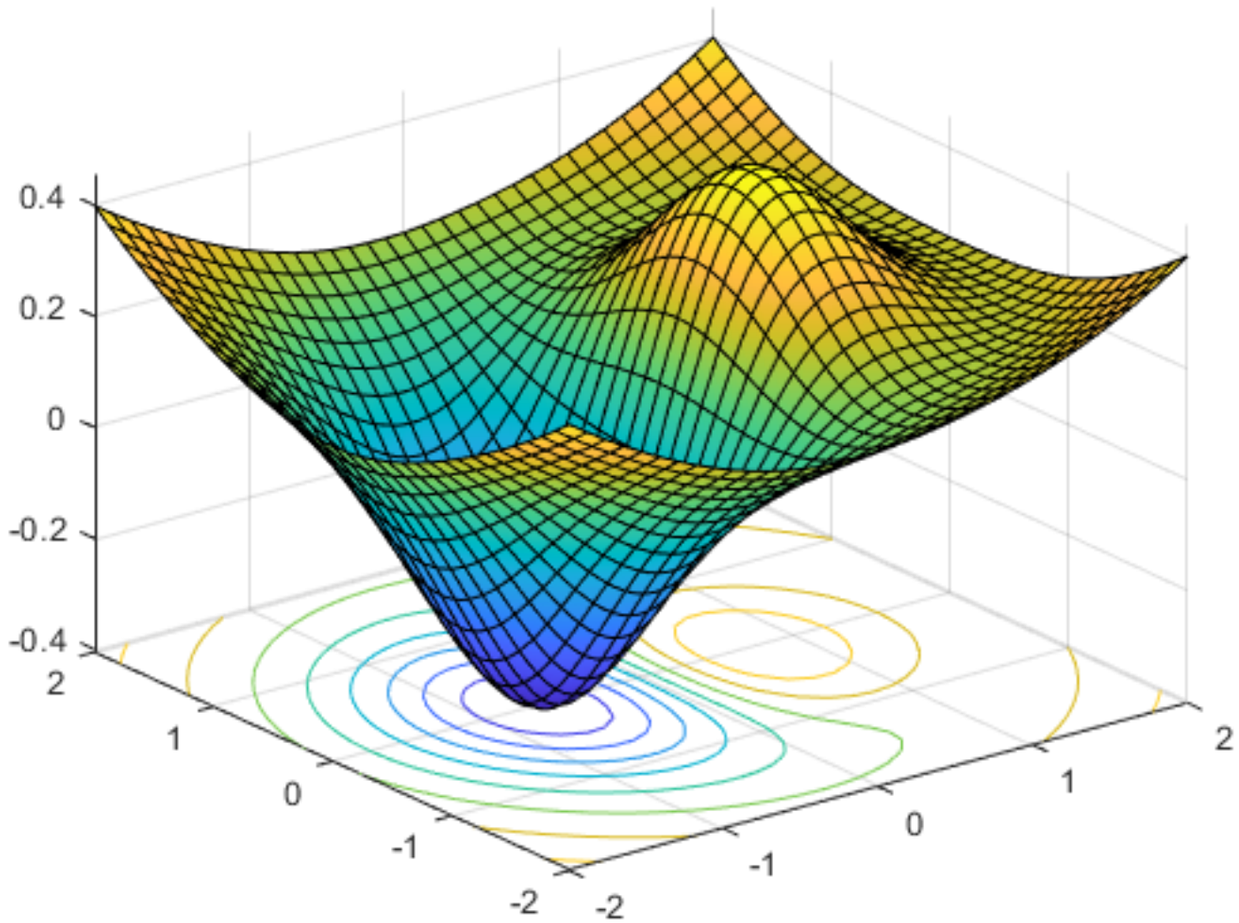
Advanced Algorithms

Lecture 23: Optimization and linear programming

Announcements

- HW 5 due on Wednesday Nov 20

Optimization



Goal: maximize or minimize a function over a “domain”

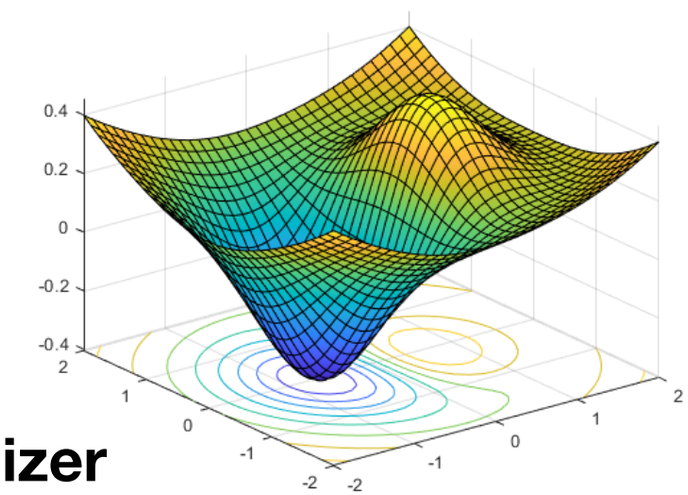
Recap

Instance of problem

Optimization formulation

variables x_1, x_2, …
constraints
objective

Optimizer



opt solution:
$x_1 = 0$, $x_2 = 1$, …

Solution to problem instance

Important aspects

- How to choose variables? (must help us obtain solution to original)
- Optimizer knows nothing about original problem or instance (only knows constraints, variables, ...) — requires proof/argument
- Important to ensure that opt problem can be solved efficiently:
  - how many variables/constraints?
  - discrete/continuous domain
  - type of constraints & objective (linear, quadratic, etc.)

Examples

- Set Cover ✓
- Matching — assigning gifts to children ✓
- Minimum spanning tree → required exp many constraints.

— What kind of constraints are allowed?

- prefer "simple" ones (OPTIMIZER's performance).

Recap: efficiency

What optimization problems can be solved efficiently?

- Linear programs — real valued variables, linear constraints, linear objective
- "Convex" programs — real valued variables, convex constraints, *minimize* convex objective
- Good heuristics for "integer linear programs" (esp binary valued), "quadratic programming", ...

Most optimizers are iterative algorithms — constantly improve solution

What about combinatorial problems?

- All reductions we've seen involve "binary" variables
- "Relaxations" of constraints

Goal: Can we solve comb. problems

Question (e.g. set cover) using linear optimization

$\downarrow$

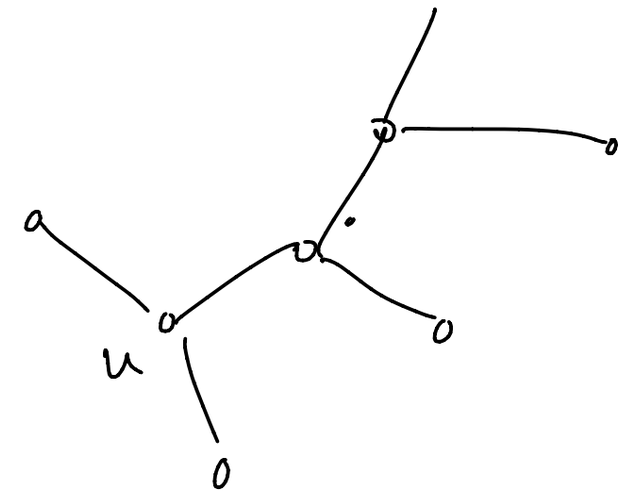
with some guarantees.

Linear programs: Continuous opt problems

Qn: Are there issues going from discrete domain $\rightarrow$ continuous?

Vertex cover relaxation

- Pick a small set of vertices s.t. every edge is "covered". (HW 3).



$x_u \in \{0,1\}$ → indicates if it's chosen.

$\forall$ edges u, v

$$x_u + x_v \geq 1$$

Objective: minimize $\sum_u x_u$.

(LP. Relaxation).

$0 \leq x_u \leq 1$

$\forall$ edges u, v

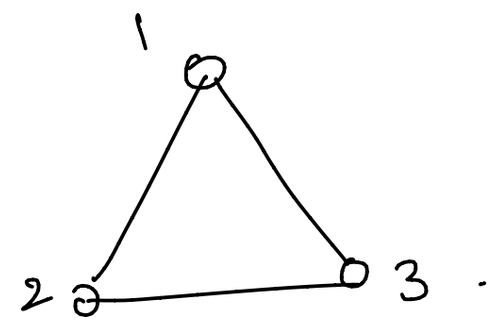
$$x_u + x_v \geq 1$$

Obj. minimize $\sum_u x_u$.

"Cheating" solutions

- Optimum solution to the LP need not have all integer values!

E.g.:



★ Integrality gap of this instance.

Size of the opt. vertex cover $= 2 \equiv$ value of original formulation.

opt Value of LP-relaxation ≤ 1.5 (because $x_1 = x_2 = x_3 = \frac{1}{2}$ satisfies all the constraints.)

$$\frac{2}{1.5} = \frac{4}{3}$$

Rounding, approximation

<u>Question:</u> Is the integrality gap potentially <u>very large</u>? (like 10?)

<u>Question:</u> Is there a way to go from a fractional solution $\{x_u\}$ to an <u>integral one</u>, without changing the objective value "<u>too much</u>"?

Claim: for vertex cover, we can convert any fractional solution $\{x_u\}$ to an feasible integer solution $\{y_u\}$, s.t.

$$\sum_u y_u \leq 2 \cdot \sum_u x_u.$$

Corollary: suppose solving the LP gave a frac. soln whose cost (obj. val.) $= \alpha$; then we can come up with int. solution with cost $\leq 2\alpha$.

"Rounding": if $x_u < 0.5$, then set $y_u = 0$

$\leq$

if $x_u \geq 0.5$, " $y_u = 1$.

<u>Feasibility</u>: if $x_u + x_v \geq 1$, then at least one of x_u, x_v is $\geq \frac{1}{2}$ $\Rightarrow$ at least one of $y_u, y_v = 1$ $\Rightarrow$ $y_u + y_v \geq 1$.

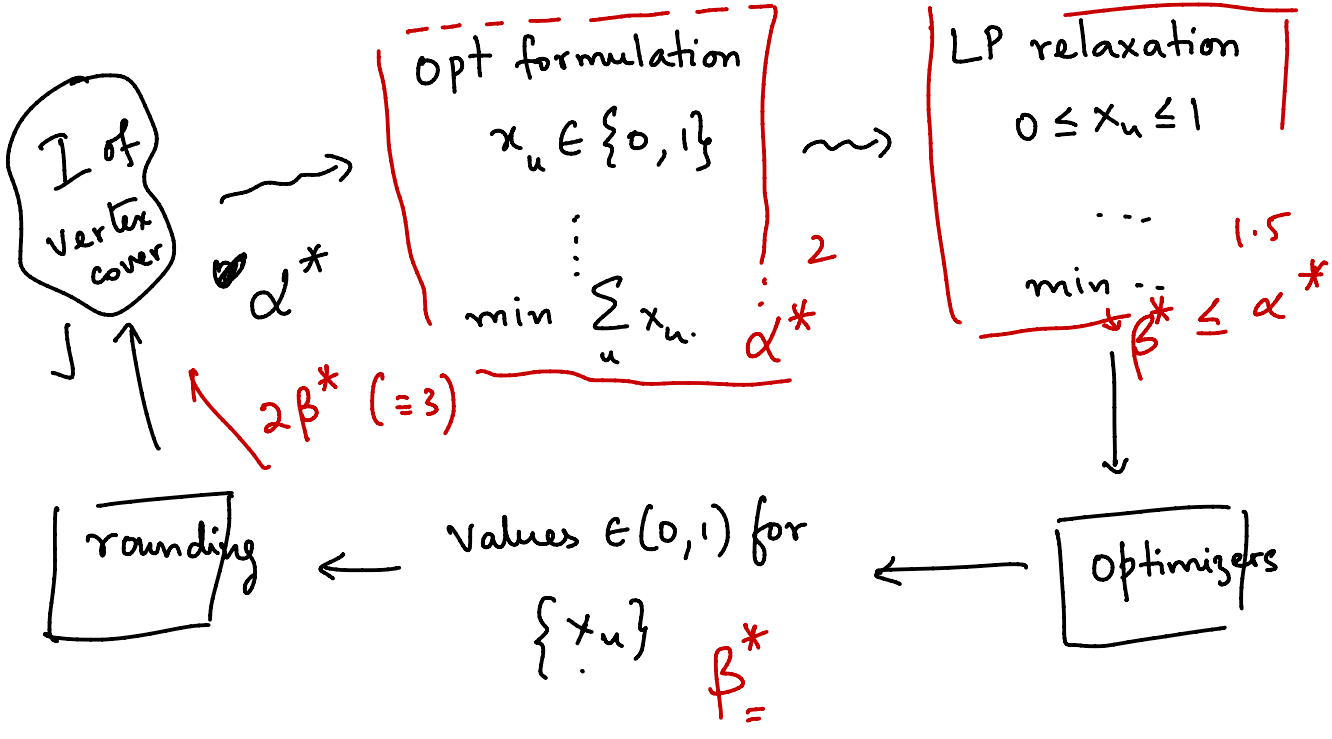
<u>Objective val</u>: for any u, rounding satisfies $y_u \leq 2x_u$

$$\sum_u y_u \leq 2 \cdot \sum_u x_u.$$

What we obtained:

"Relax-&-Round" paradigm for designing approx algos.

Factor-2 approximation algorithm



Set cover

- What are variables?
- Constraints?
- Objective?

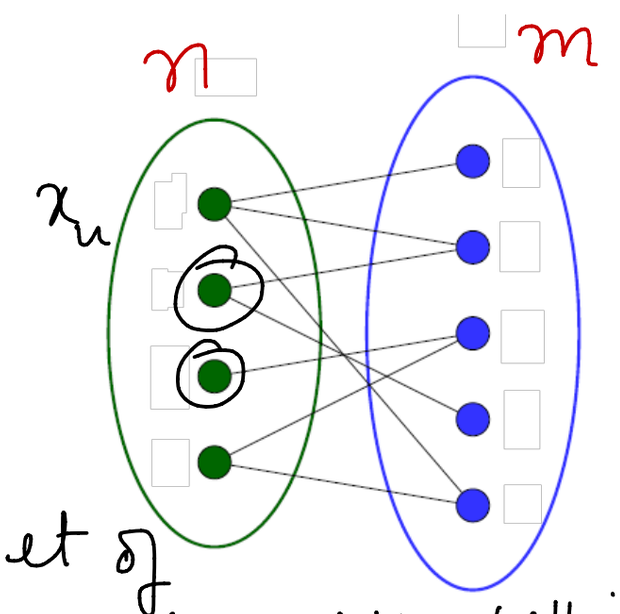
Problem: suppose we have *n* people, and *m* "desired skills"; each person has a subset of the skills. Pick the smallest subset of people such that every skill is *covered*

<u>Variables:</u> $x_u \in \{0,1\}$.

<u>Constraints:</u> for each skill j,

$$\sum_{u \in S_j} x_u \geq 1,$$ where S_j: set of people with skill j

$$\min \sum_u x_u.$$



OPT formulation

$x_u \in \{0,1\}$ for $u \in [n]$

$\forall j \in [m], \quad \sum_{u \in S_j} x_u \geq 1$

$\min \sum_u x_u .$

☆ LP formulation

$0 \leq x_u \leq 1$

$\forall j \in [m], \quad \sum_{u \in S_j} x_u \geq 1$

$\min \sum_u x_u .$

<u>Qn:</u> will the same "rounding" work?

$$y_u = \begin{cases} 1 & \text{if } x_u \geq \frac{1}{2} \\ 0 & \text{o/wise} \end{cases} ; \quad \text{is } \{y_u\} \text{ feasible?}$$

$\downarrow$

No! $x_1 + x_2 + x_3 \geq 1$ can happen but none of $y_i = 1$

Set cover “polytope”

Set cover – rounding

<u>Randomized rounding</u>

$x_u = 0.1$

$x_v = 0.6$; $x_w = 0.8$

→ Suppose we round every $x_u \to 1$ w.p. equal to x_u.

$$\underbrace{\frac{1}{100} + \frac{1}{100} + \cdots + \frac{1}{100}}_{100}$$

$$\underset{\frac{1}{3}}{x_1} + \underset{\frac{1}{3}}{x_2} + \underset{\frac{1}{3}}{x_3} \geq 1$$

In expectation, one of the three gets rounded up to 1!

Let α be a parameter (TBD), & suppose we

Set $y_u = \begin{cases} 1 & \text{w.p. } \alpha x_u \to \min(1, \alpha x_u). \\ 0 & \text{o/wise.} \end{cases}$

Suppose $x_1 + x_2 + \dots + x_t \geq 1$

what is the prob. that $y_1 + \dots + y_t = 0$?

$1 - a \leq e^{-a}$

$$= (1 - \alpha x_1)(1 - \alpha x_2) \dots (1 - \alpha x_t)$$

$$\leq e^{-\alpha x_1} \cdot e^{-\alpha x_2} \dots e^{-\alpha x_t} = e^{-\alpha\left(\sum_{i=1}^{t} x_i\right)}$$

$\leq e^{-\alpha}$