Advanced Algorithms

Lecture 27: Reductions, Review

Basic complexity classes

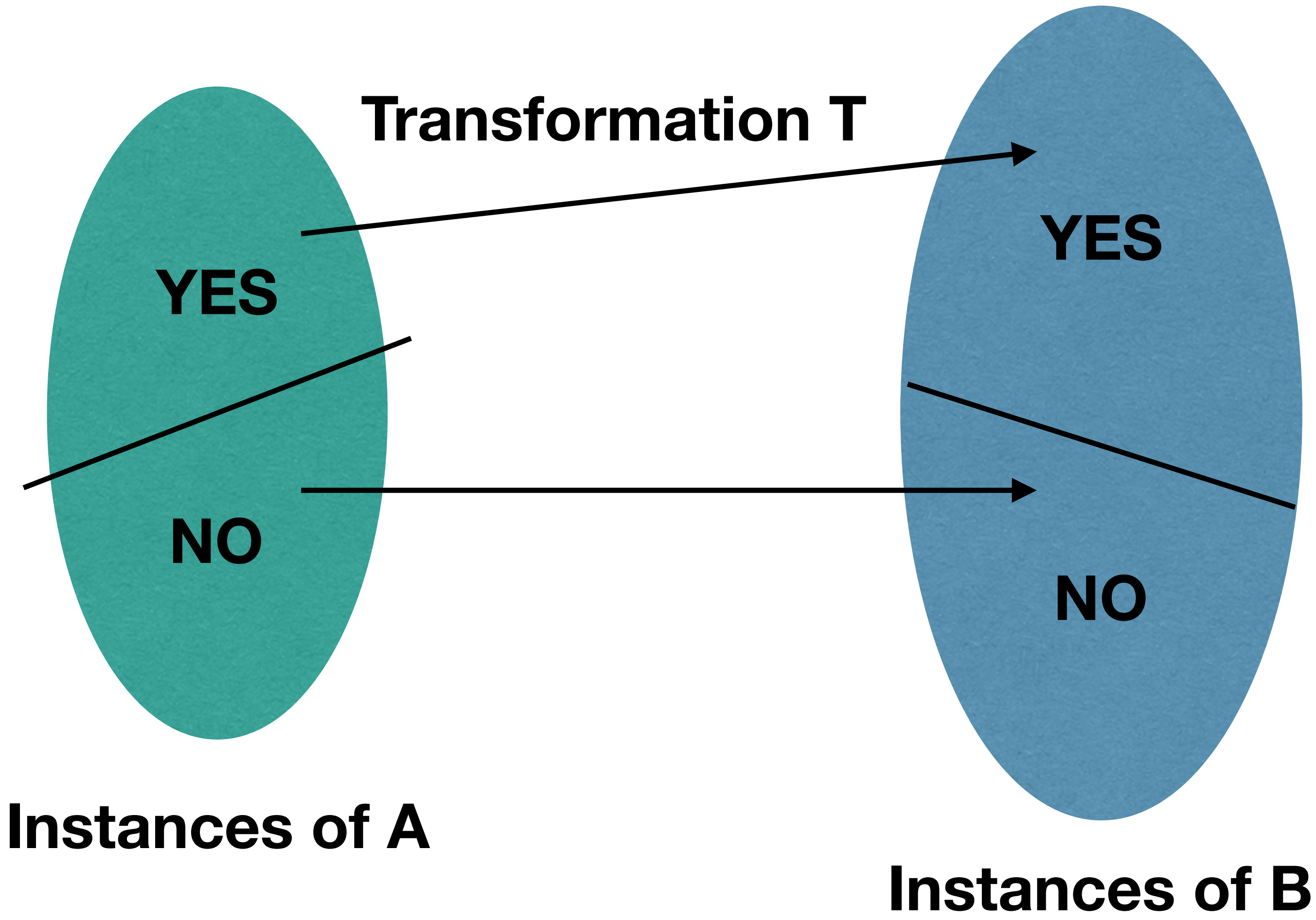
P: set of all decision problems that can be solved in polynomial time

NP: the set of all decision problems for which there is a polynomial time “verification algorithm”

Reductions between problems

"I can't find an efficient algorithm, but neither can all these famous people."

Definition. a <u>poly-time reduction</u> from problem A to problem B is a transformation that maps instances of A to instances of B s.t. YES instances map to YES ones and NO instances map to NO ones.



- T runs in polynomial time
- T has no idea if instance is YES or NO while transforming
- T can be applied to any instance of A, obtaining an instance of B

NP-hard and NP-complete

- **NP-hard:** a problem *A* is said to be NP-hard if <u>Boolean SAT</u> reduces to *A* (can replace with Independent Set)

- **NP-complete:** a problem *A* is said to be NP-complete if Boolean SAT reduces to *A* <u>**and**</u> *A* reduces to Boolean SAT

Boolean satisfiability (SAT)

Problem. given a boolean circuit (AND/OR/NOT gates) with inputs $x_1, x_2, \ldots, x_n$ and output $f(x_1, x_2, \ldots, x_n)$, does there exist a T/F assignment of inputs for which f is TRUE?

Easy to see: SAT is in NP [Witness is the assignment, verification algorithm is the circuit itself!]

Cook-Levin theorem

Theorem. any problem in NP (i.e., any problem with a poly time verification oracle) has a poly time reduction to Boolean SAT.

Boolean SAT is the "hardest" problem in NP

3-SAT

Problem. given a boolean function $f(x_1, x_2, \ldots, x_n)$ of <u>the following type</u>: $f(x_1, x_2, \ldots, x_n) = C_1 \wedge C_2 \wedge \ldots \wedge C_m$, where each C_i has the form $C_i = l_{i_1} \vee l_{i_2} \vee l_{i_3}$ for some "literals" (variable or negation), decide if there is an assignment to the x_i such that f is TRUE

Is a special case of SAT, so "easier" than SAT

Turns out that SAT $\leq_p$ 3-SAT as well

Independent Set (IS)

Problem. given a graph $G = (V, E)$ and parameter k, find if there is an independent set of size k

IS $\in$ NP (because there's a verification procedure)

3-SAT $\leq_p$ IndependentSet

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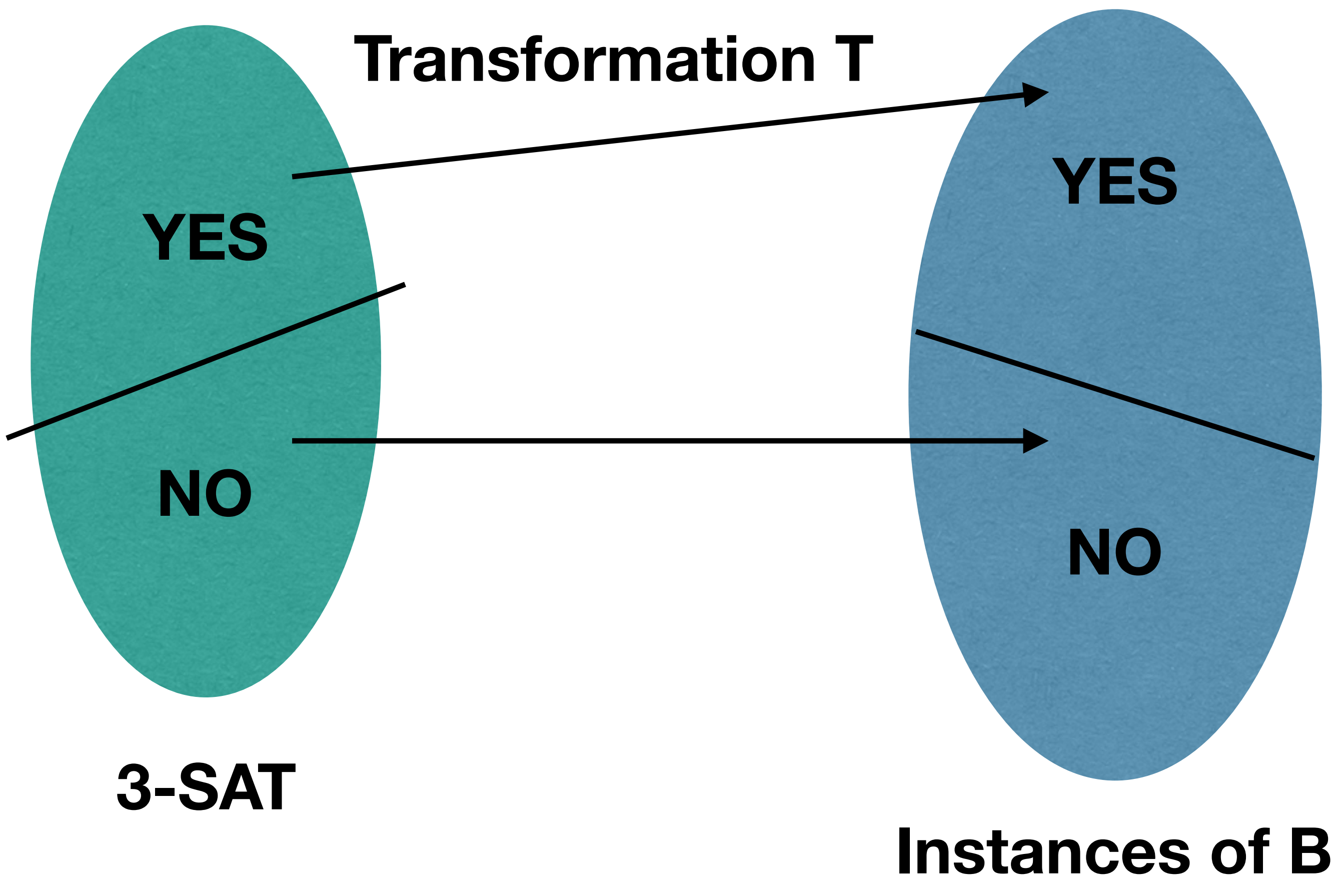
3-SAT reduces to IS => SAT reduces to IS => IS is NP-hard

Hardness of approximation

- Can problems be NP-hard to solve approximately? (e.g., set cover)
- **What kind of reduction?**

Hardness of approximation



"Standard" NP hardness

Hardness of approximation

- "PCP theorem": there is a "gap-inducing" reduction from SAT to 3-SAT
- Used to show hardness of approximating various other problems

Course review