Advanced Algorithms

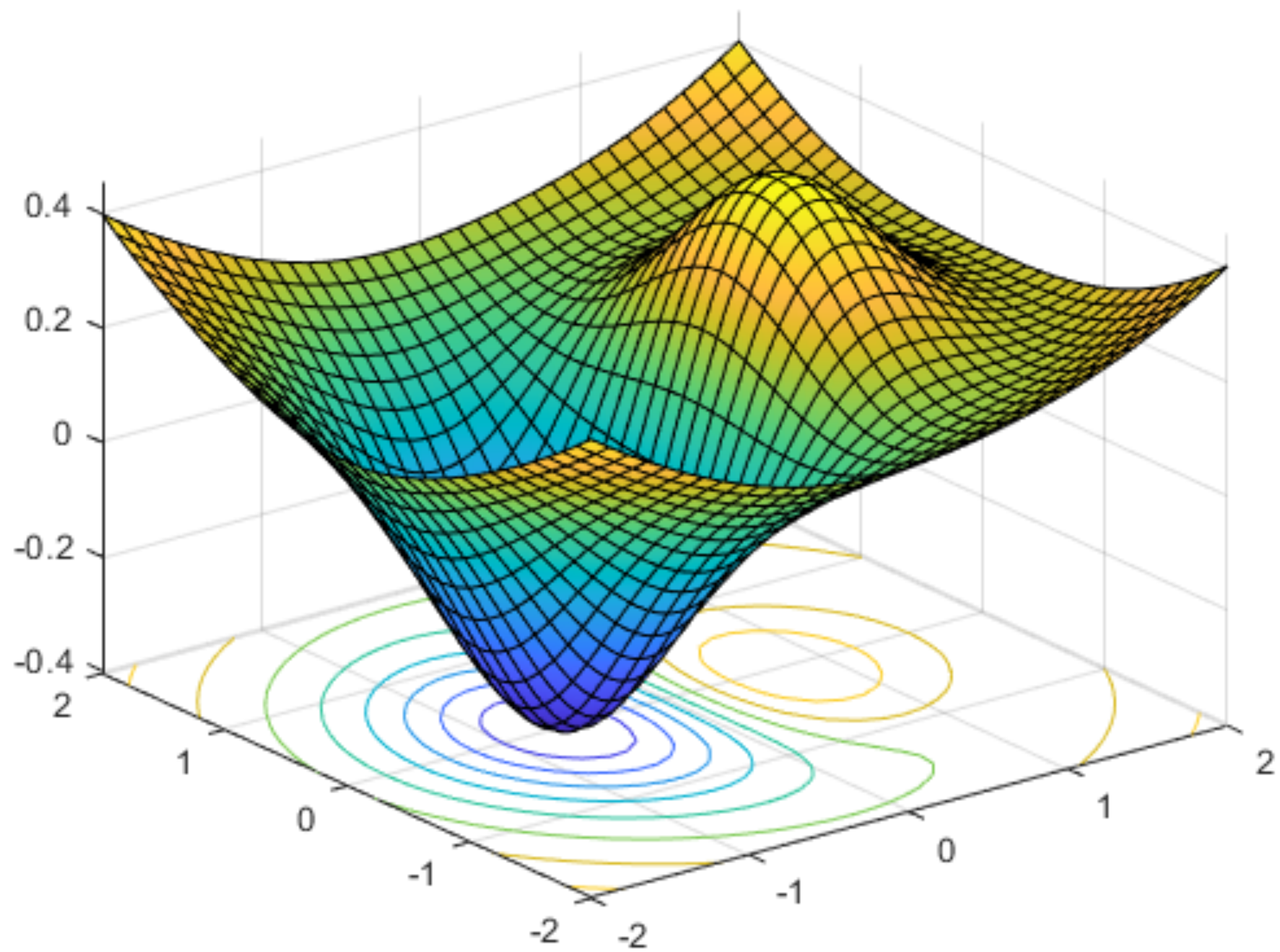
Lecture 20: Optimization

Announcements

- HW 5 — will be out today, due Wednesday Nov 20
- Next Tuesday (Nov 12): Guest lecture by Suresh Venkat on **Algorithmic Fairness**

Optimization formulations

Optimization?



Typical goal: maximize or minimize a function over a "domain"

Optimization – formal

$x_1, x_2, \ldots, x_n \rightarrow$ real valued

discrete valued: $\{0, 1\}$.

- Variables
- Objective function (needs to be optimized) $f(x_1, x_2, \ldots, x_n)$
- Constraints that define domain

$$0 \le x_i \le 1 \quad ; \quad \sum_i x_i^2 = 1$$

Typical issues

- Can optimum be found efficiently? (can we maximize/min..)
- Local vs global optima
- Algorithms for optimization (first/second order methods, descent techniques, ...)

Topics of another course… we will use opt as a blackbox

Variables: $\{x_i\}_{i=1}^{n}$

<u>Obj</u>: min $\|Vx - y\|^2$

Classic examples

- Unconstrained optimization — regression

"observations" $\in \mathbb{R}^n$ value $\in \mathbb{R}$

$$V = \begin{bmatrix} \text{——} \; v_1 \; \text{——} \\ \text{——} \; v_2 \; \text{——} \\ \vdots \\ \text{——} \; v_m \; \text{——} \end{bmatrix} \quad \begin{matrix} \rightsquigarrow y_1 \\ \rightsquigarrow y_2 \\ \\ \rightsquigarrow y_m \end{matrix}$$

(v_i, y_i)

$f(v_i) = y_i$

$\equiv$

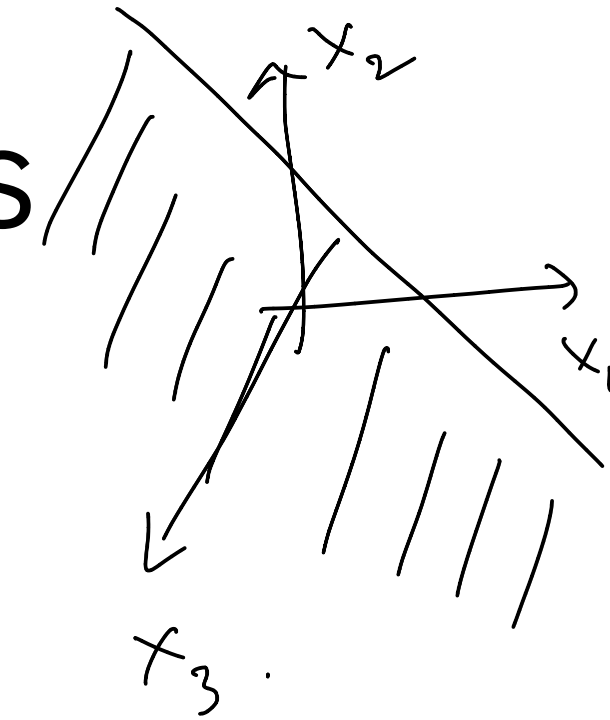
$\langle v_i, x \rangle$

$\in \mathbb{R}^m$

$\| \overbrace{Vx} - y \|^2$ is minimized.

Goal is to find $x \in \mathbb{R}^n$ s.t. $Vx \approx y$

Classic examples



- Linear optimization (or linear *programming*)

Variables: $x_1, \ldots, x_n \to$ real valued

Objective: $\max \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$, where c_i are some coeffs.

Constraints:

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1$$

$$a_{21} x_1 + \cdots + a_{2n} x_n \le b_2$$

$$\vdots$$

Discrete problems as OPT

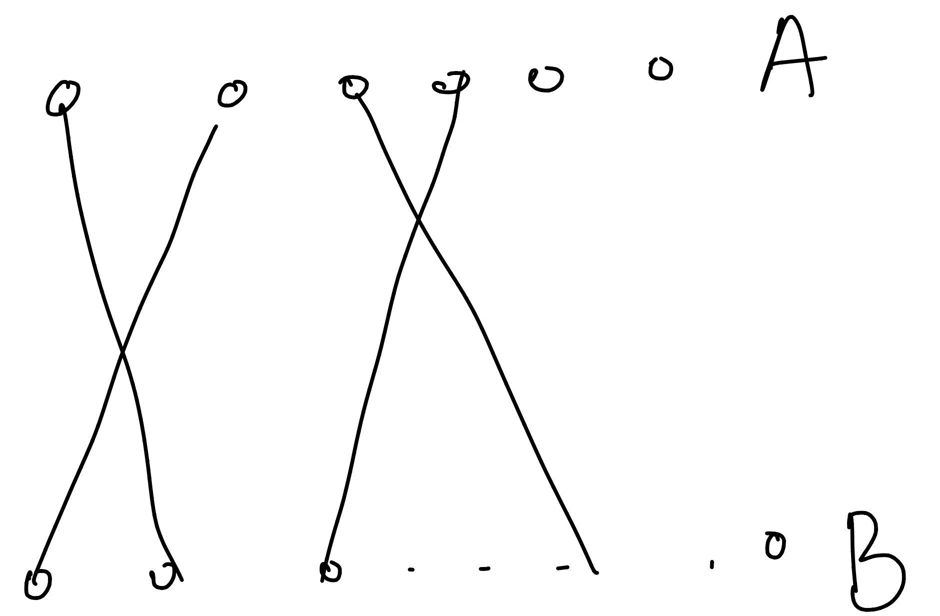
Goal: express "combinatorial" problems as OPT, ...

- Matching in bipartite graphs [blast from the past — children, gifts, ...]
- Set cover [choose smallest set of people to "cover" all skills
- Minimum spanning tree
- Shortest path?

Bipartite .

Matching problem

Problem: suppose we have *n* children and *n* gifts. Each child has some "happiness value" (V_{ij}) for each gift. Find an allocation (one gift per child) so that total happiness is maximized.



Bipartite matching as opt

- What are variables?
- Constraints?
- Objective?

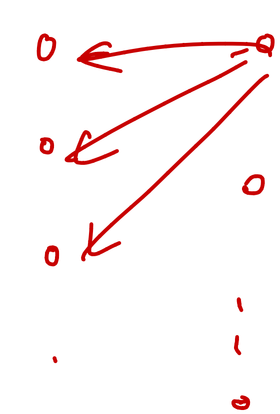
Candidate 1: y_i : index of gift that child i gets.

Candidate 2: have "boolean" variables: $\boxed{x_{ij} \in \{0,1\}}$.

supposed to ~~ded~~ denote if child i received gift j (or not).

$$\begin{cases} \text{for every child } i, \quad \sum_j x_{ij} = 1 \; ; \quad \text{each } x_{ij} \in \{0,1\} \\ \text{for every gift } j, \quad \sum_i x_{ij} = 1 \end{cases}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$



Objective value? total happiness val. of our assignment.

$$\text{max} \sum_{\text{all } i,j} V_{ij} \, x_{ij}$$

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

$$\parallel\!\!\!\!\!\!\; \backslash$$

"ideally": $\sum_{\text{children } i} V_{i, \text{gift}(i)}$, where gift(i) is the gift given to child i.

Optimization paradigm

Problem

matching

→

Opt problem

var. $x_{ij} \in \{0,1\}$

constraints: ... 2n ...

objective: max $\sum_{i,j} v_{ij} x_{ij}$

→ "oracle".

→ Values for x_{ij}

→ Solution to original problem.

Correctness of paradigm rests on

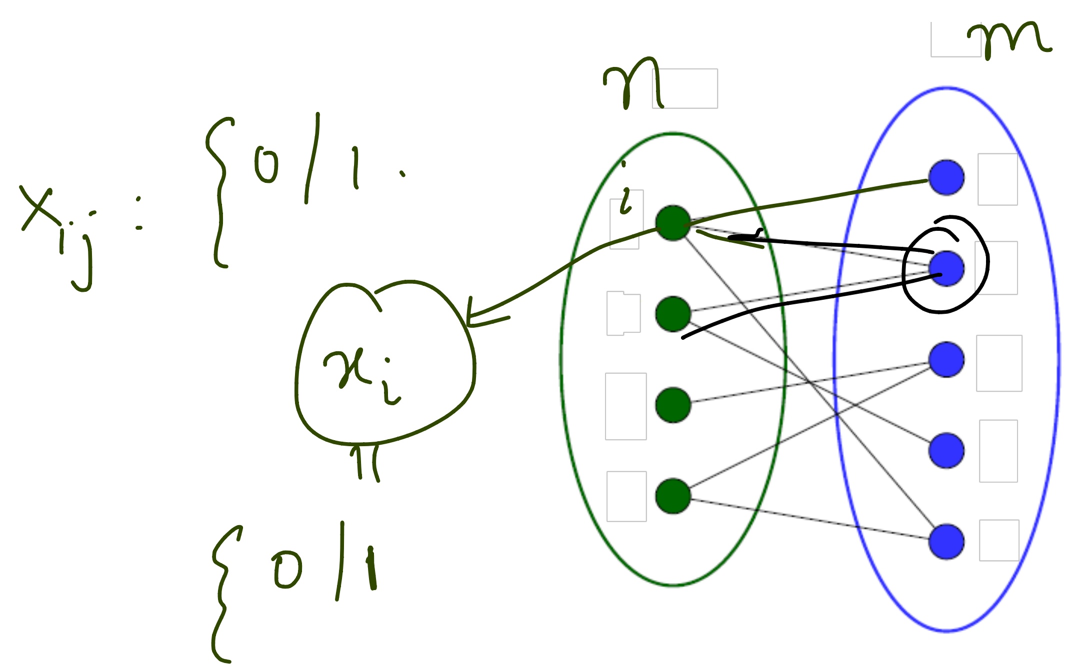
(requires some proof.).

$n^3, \ldots$

Set cover

- What are variables?
- Constraints?
- Objective?

Problem: suppose we have *n* people, and *m* "desired skills"; each person has a subset of the skills. Pick the smallest subset of people such that every skill is *covered*



Set cover as opt

- What are variables? : x_i for each person i; supposed to be 1 if i is chosen in the set & 0 o/wise
- Constraints?
- Objective?

have "m" constraints

for each skill j, denoting by S_j the set of people with that skill, $\sum_{i \in S_j} x_i \geq 1$

Objective: minimize $(x_1 + x_2 + \dots + x_n)$

Spanning tree

$O((m+n)\log n)$.

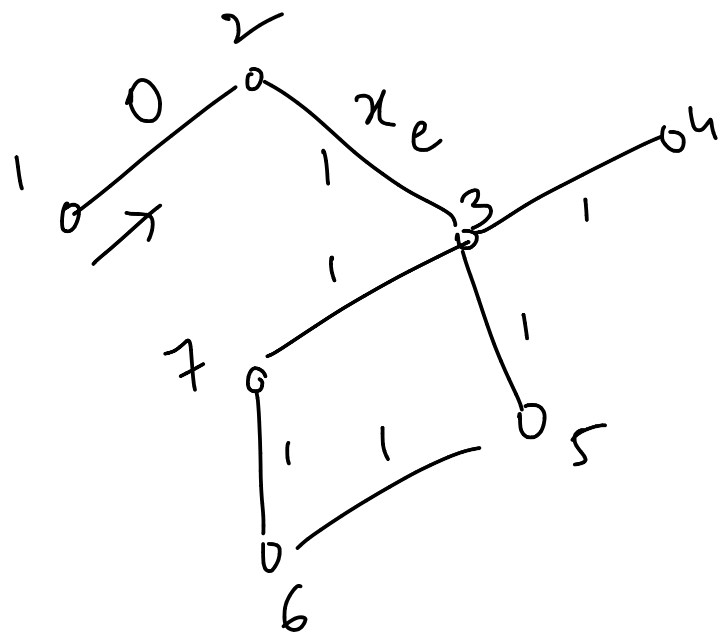
Problem: let G = (V, E) be a (simple, undirected) graph with edge weights $\{w_e\}$ (>0). Pick a subset of the edges, such that (a) all vertices are "connected", (b) total weight of edges is minimized

- What are variables? $\longrightarrow$ x_e for one every edge 'e'. (supposed to be binary).
- Constraints?
- Objective?

$$\longrightarrow \sum_{\substack{\text{all } e \\ \text{s.t. } x_e=1}} w_e \equiv \underline{\underline{\sum_{\text{all } e} x_e w_e}}.$$

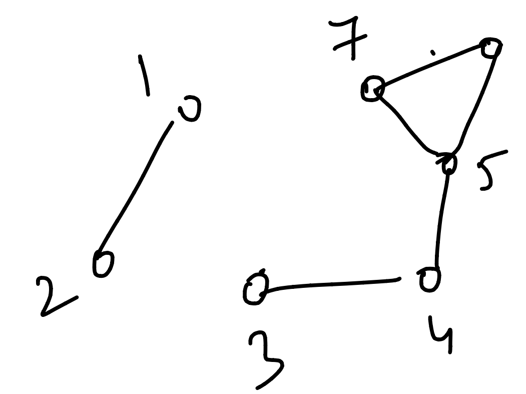
Constraints for MST

$x_e^2 = x_e$.



One constraint: $\sum_{\text{all edges } e} x_e = n-1$.

another constraint:



for every i, $\sum_{\text{edges } e \text{ adjacent to } i} x_e \geq 1$

☆ have a constraint for every cycle: let $e_1, e_2, \ldots, e_r$ be the edges; then $x_{e_1} + x_{e_2} + \cdots + x_{e_r} \leq r-1$.

Why express as opt?

- Many algorithms, heuristics
- Often “hard” to guarantee good solution
- *Relaxations* can still be useful