Homework 3
Spring 2019 Solutions

Exercise 1. 40 pts

We can use Shannon's expansion with respect to two varibles inorder to derive the circuit implmentation using only 4-to-1 multiplexors. The general format for this expansion is as follows:

$$f(x_3, x_2, x_1, x_0) \quad = \quad \overline{x}_1\overline{x}_0 f_{\overline{x}_1\overline{x}_0} + \overline{x}_1 x_0 f_{\overline{x}_1 x_0} + x_1\overline{x}_0 f_{x_1\overline{x}_0} + x_1 x_0 f_{x_1 x_0}$$

The resulting 4-to-1 multiplexor would then have the following format:

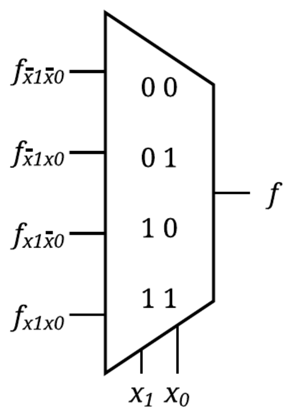

Figure 1: Generalized 4-to-1 mux

Using this template, we can solve for the four given functions.

$$\begin{aligned}
\text{sum}_1 &= a_1 + b_1 + a_0 b_0 \\
&= \overline{a}_1\overline{a}_0 sum_{1\overline{a}_1\overline{a}_0} + \overline{a}_1 a_0 sum_{1\overline{a}_1 a_0} + a_1\overline{a}_0 sum_{1 a_1\overline{a}_0} + a_1 a_0 sum_{1 a_1 a_0} \\
&= \overline{a}_1\overline{a}_0(b_1) + \overline{a}_1 a_0(b_0 + b_1) + a_1\overline{a}_0(1) + a_1 a_0(1)
\end{aligned}$$

We have to perform Shannon's expansion again, since the third cofactor is not a constant and contains more than one cube. Since we are implementing with 4-to-1 multiplexors only, we need to expand with respect to the remaining two variables.

$$
\begin{aligned}
g &= b_0 + b_1 \\
&= \overline{b}_1\overline{b}_0 g_{\overline{b}_1\overline{b}_0} + \overline{b}_1 b_0 g_{\overline{b}_1 b_0} + b_1\overline{b}_0 g_{b_1\overline{b}_0} + b_1 b_0 g_{b_1 b_0} \\
&= \overline{b}_1\overline{b}_0(0) + \overline{b}_1 b_0(1) + b_1\overline{b}_0(1) + b_1 b_0(1)
\end{aligned}
$$

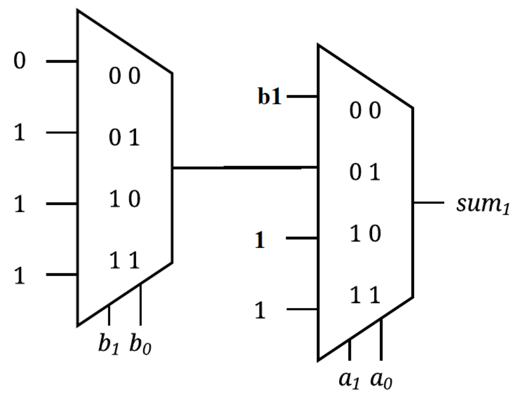

Figure 2: Implementation of sum_1 using only 4-to-1 multiplexors

The next equation only contains two variables, so we will expand using the two variables.

$$
\begin{aligned}
\text{sum}_0 &= \overline{a}_0 b_0 + a_0\overline{b}_0 \\
&= \overline{a}_0\overline{b}_0 sum_{0\overline{a}_0\overline{b}_0} + \overline{a}_0 b_0 sum_{0\overline{a}_0 b_0} + a_0\overline{b}_0 sum_{0a_0\overline{b}_0} + a_1 b_0 sum_{0a_0 b_0} \\
&= \overline{a}_0\overline{b}_0(0) + \overline{a}_0 b_0(1) + a_0\overline{b}_0(1) + a_0 b_0(0)
\end{aligned}
$$

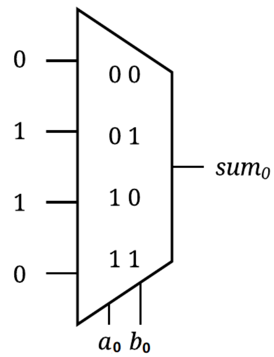

Figure 3: Implementation of sum_0 using only 4-to-1 multiplexors

$$
\begin{aligned}
\text{diff}_1 &= a_1 a_0 \overline{b_1} + a_1 \overline{a_0}\,\overline{b_1}\ \overline{b_0} \\
&= \overline{a}_0 \overline{b}_0 diff_{1\overline{a}_0\overline{b}_0} + \overline{a}_0 b_0 diff_{1\overline{a}_0 b_0} + a_0 \overline{b}_0 diff_{1a_0\overline{b}_0} + a_0 b_0 diff_{1a_0b_0} \\
&= \overline{a}_0 \overline{b}_0 (a_1\overline{b_1}) + \overline{a}_0 b_0 (0) + a_0 \overline{b}_0 (a_1\overline{b_1}) + a_0 b_0 (a_1\overline{b_1})
\end{aligned}
$$

The fourth expression needs to be expanded further.

$$
\begin{aligned}
g &= a_1\overline{b_1} \\
&= \overline{a}_1 \overline{b}_1 g_{\overline{a}_1\overline{b}_1} + \overline{a}_1 b_1 g_{\overline{a}_1 b_1} + a_1 \overline{b}_1 g_{a_1\overline{b}_1} + a_1 b_1 g_{a_1b_1} \\
&= \overline{a}_1 \overline{b}_1 (0) + \overline{a}_1 b_1 (0) + a_1 \overline{b}_1 (1) + a_1 b_1 (0)
\end{aligned}
$$

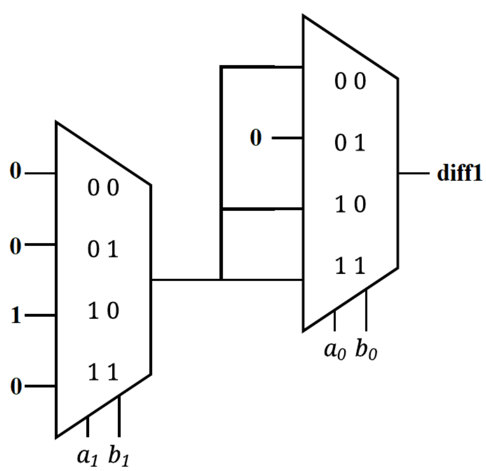

Figure 4: Implementation of diff_1 using only 4-to-1 multiplexors

The implementation of the last function is next.

$$\begin{aligned}
\text{diff}_0 &= a_0\overline{b_0} + \overline{a_0}b_0 \\
&= \overline{a}_0\overline{b}_0 diff_{0\overline{a}_0\overline{b}_0} + \overline{a}_0 b_0 diff_{0\overline{a}_0 b_0} + a_0\overline{b}_0 diff_{0a_0\overline{b}_0} + a_0 b_0 diff_{0a_0b_0} \\
&= \overline{a}_0\overline{b}_0(0) + \overline{a}_0 b_0(1) + a_0\overline{b}_0(1) + a_0 b_0(0)
\end{aligned}$$

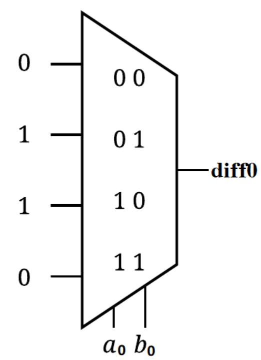

Figure 5: Implementation of diff_0 using only 4-to-1 multiplexors

Exercise 2. 15 pts

Simply take Shannon's expansion with respect to each literal in the equation. We have four literals (a_0, b_0, a_1, b_1), so we need to take Shannon's expansion four times to derive the set minterms covererd by $diff_0$.

$$\begin{aligned}
\text{diff}_0 &= a_0\bar{b_0} + \bar{a_0}b_0 \\
&= a_1(a_0\bar{b_0} + \bar{a_0}b_0) + \bar{a_1}(a_0\bar{b_0} + \bar{a_0}b_0) && \text{Shannon's on } a_1 \\
&= a_0\bar{b_0}a_1 + \bar{a_0}b_0a_1 + a_0\bar{b_0}\bar{a_1} + \bar{a_0}b_0\bar{a_1} && \text{Distributive} \\
&= b_1(a_0\bar{b_0}a_1 + \bar{a_0}b_0a_1 + a_0\bar{b_0}\bar{a_1} + \bar{a_0}b_0\bar{a_1}) + \\
&\quad \bar{b_1}(a_0\bar{b_0}a_1 + \bar{a_0}b_0a_1 + a_0\bar{b_0}\bar{a_1} + \bar{a_0}b_0\bar{a_1}) && \text{Shannon's on } b_1 \\
&= a_0\bar{b_0}a_1b_1 + \bar{a_0}b_0a_1b_1 + a_0\bar{b_0}\bar{a_1}b_1 + \bar{a_0}b_0\bar{a_1}b_1 + a_0\bar{b_0}a_1\bar{b_1} + \bar{a_0}b_0a_1\bar{b_1} \\
&\quad + a_0\bar{b_0}\bar{a_1}\bar{b_1} + \bar{a_0}b_0\bar{a_1}\bar{b_1} && \text{Distributive}
\end{aligned}$$

To verify our result, here is a table showing all of the possible inputs to diff_0, the output of diff_0, and the specific minterm in our equation which covers that particular output.

$a_0b_0a_1b_1$	diff_0	minterms
0000	0	N/A
0001	0	N/A
0010	0	N/A
0011	0	N/A
0100	1	$\bar{a}_0b_0\bar{a}_1\bar{b}_1$
0101	1	$\bar{a}_0b_0\bar{a}_1b_1$
0110	1	$\bar{a}_0b_0a_1\bar{b}_1$
0111	1	$\bar{a}_0b_0a_1b_1$
1000	1	$a_0\bar{b}_0\bar{a}_1\bar{b}_1$
1001	1	$a_0\bar{b}_0\bar{a}_1b_1$
1010	1	$a_0\bar{b}_0a_1\bar{b}_1$
1011	1	$a_0\bar{b}_0a_1b_1$
1100	0	N/A
1101	0	N/A
1110	0	N/A
1111	0	N/A

Exercise 3. 15 pts
It is tempting to think that the dual of f_w is $f_{\bar{w}}$, but this is not true. For example, let $f(w,x) = wx$. Then $f_w = x$ and the dual of f_w is still x whereas $f_{\bar{w}} = 0$. Furtheremore, the notion of just taking one dual to $f = wf_w + \bar{w}f_{\bar{w}}$ is not correct since the dual of the right hand side is equal to the dual of f and not f. So here is a correct couple of proofs. The first directly uses dualization and the second essentially uses it in the form of DeMorgan's law.

The first proof. For notational purposes let a superscript d denote the dual of a function. For example, f^d is the dual of f and $(f_w)^d$ is the dual of f_w. For simplicity, write $(f_w)^d$ as f_w^d. Then, the dual of $f = wf_w + \bar{w}f_{\bar{w}}$ is

$$f^d = (w + f_w^d)(\bar{w} + f_{\bar{w}}^d).$$

Using the distributive law (and commutativity) on the right hand side gives

$$f^d = w\bar{w} + wf^d\bar{w} + \bar{w}f_w^d + f_w^d f_{\bar{w}}^d = wf^d\bar{w} + \bar{w}f_w^d + f_w^d f_{\bar{w}}^d.$$

After using dualization again and noting that $(f^d)^d = f$ (that is, taking the dual twice gets back the original function), one gets

$$f = (w + f_{\bar{w}})(\bar{w} + f_w)(f_w + f_{\bar{w}})$$

which yields

$$f = (w + f_{\bar{w}})(\bar{w} + f_w)(f_w + f_{\bar{w}})$$

essentially by consensus.

Now the second proof is like the first, but uses the complement so as not to go into dual land. It works like this

$$
\begin{aligned}
f &= w f_w + \bar{w} f_{\bar{w}} \\
&= \overline{\overline{w f_w + \bar{w} f_{\bar{w}}}} && \text{rule 9} \\
&= \overline{(\bar{w} + \bar{f}_w)(w + \bar{f}_{\bar{w}})} && \text{DeMorgan} \\
&= \overline{\bar{w} w + \bar{w} \bar{f}_{\bar{w}} + w \bar{f}_w + \bar{f}_w \bar{f}_{\bar{w}}} && \text{distributive law 12 a} \\
&= \overline{\bar{w} \bar{f} \bar{w} + w \bar{f}_w + \bar{f}_w \bar{f}_{\bar{w}}} && \text{6b,8a} \\
&= (w + f_{\bar{w}})(\bar{w} + f_w)(f_w + f_{\bar{w}}) && \text{DeMorgan} \\
&= (w + f_{\bar{w}})(\bar{w} + f_w) && \text{Consensus 17b}
\end{aligned}
$$

Exercise 4. 30 pts

Verilog solution is not included to avoid explicit copying of code.