Asynchronous Circuit Design

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Lecture 4: Graphical Representations
Chapter 4

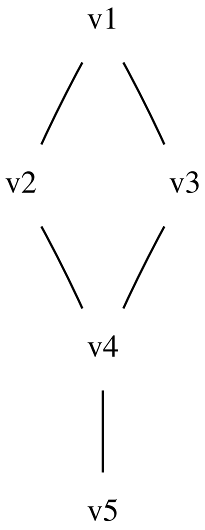
Chapter Overview

- HDL's allow specification of large systems.
- Graphs allow pictorial representation of small examples, and they are used by virtually every CAD algorithm.
- The chapter discusses the following types of graphs:
    - State machines
    - Petri-nets

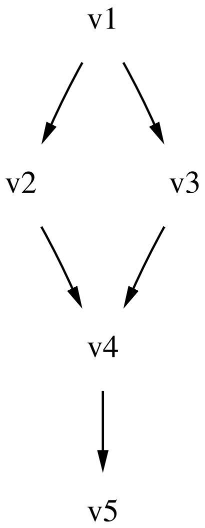
Graph Basics

- A graph G is composed of a finite nonempty set of *vertices* V and a binary relation, R ($R \subseteq V \times V$).
- *Undirected graphs*:
  - R is an irreflexive symmetric relation.
  - Since R is symmetric, $(u, v) \in R \Rightarrow (v, u) \in R$.
  - E is the set of symmetric pairs, or *edges* (denoted uv).
- *Directed graphs*, or *digraphs*:
  - R does not need to be either irreflexive or symmetric.
  - E is the set of *directed edges* or *arcs* (denoted (u,v)).

A Simple Graph



A Simple Directed Graph



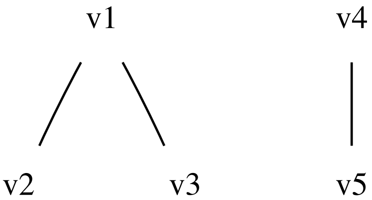
Additional Graph Definitions

- $|V|$ is called the *order* of G.
- $|E|$ is called the *size* of G.
- $V(G)$ and $E(G)$ are the vertex and edge sets for G.
- If $e = (u, v) \in E(G)$, e *joins* u and v.
- If $e = (u, v) \in E(G)$, u and v are *incident* with e.
- If $(u, v) \in E(G)$, v is *adjacent* to u.
- If $(u, v) \notin E(G)$, u and v are *nonadjacent* vertices.

Connected Graphs

- *u*-*v* *path* is an alternating sequence of vertices and edges beginning with *u* and ending with *v*.
- The length of a *u*-*v* path is the number of edges in the path.
- If there exists a *u*-*v* path, then *v* is *reachable* from *u*.
- A *u*-*v* path is *simple* if it does not repeat any vertex.
- If for every pair of vertices *u* and *v* there exists a *u*-*v* path, the graph is connected.

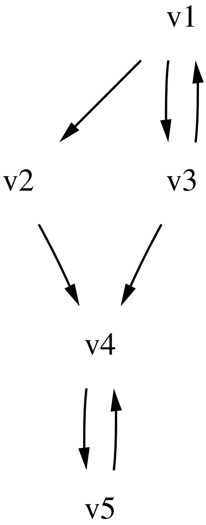
A Unconnected Graph



Directed Acyclic Graphs

- In a digraph, a u-v path forms a *cycle* if $u = v$.
- If the u-v path excluding u is simple, then the cycle is *simple*.
- A cycle of length 1 is a self-loop.
- A digraph with no self-loops is *simple*.
- In an undirected graph, a u-v path is a cycle only if simple.
- A graph which contains no cycles is *acyclic*.
- An acyclic digraph is called a *directed acyclic graph* or *DAG*.

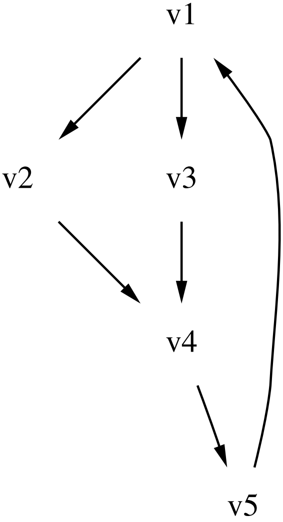
A Cyclic Digraph



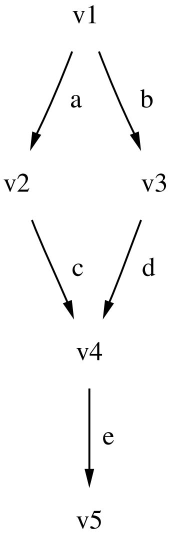
More Graph Properties

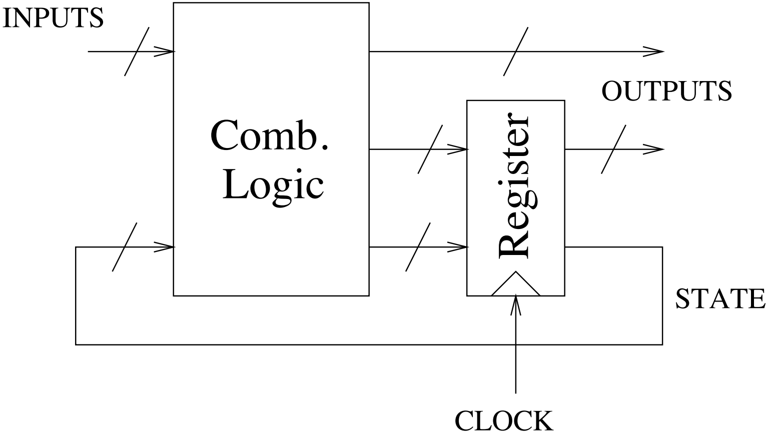
- A *digraph* G is *strongly connected* if for every two distinct vertices u and v, there exists a u-v path and a v-u path.
- A graph is *bipartite* if there exists a partition of V into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with V_2.
- A labeled graph is a triple $\langle V, R, L\rangle$ in which L is a labeling function associated either to the set of vertices or edges.

A Strongly Connected Digraph

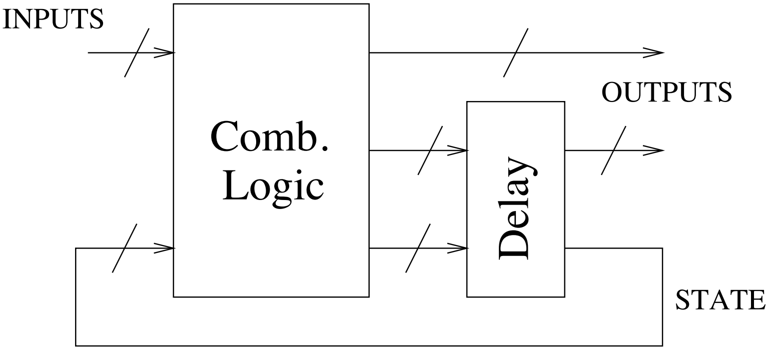


A Simple Labeled Directed Graph




INPUTS
Comb. Logic
Register
OUTPUTS
STATE
CLOCK

An Asynchronous FSM



Finite State Machines

- I is the input alphabet;
- O is the output alphabet;
- S is the finite, non-empty set of states;
- $S_0 \subseteq S$ is the set of initial (reset) states;
- $\delta : S \times I \longrightarrow S$ is the next-state function;
- $\lambda : S \times I \longrightarrow O$ is the output function for a *Mealy* machine (or $\lambda : S \longrightarrow O$ for a *Moore* machine).

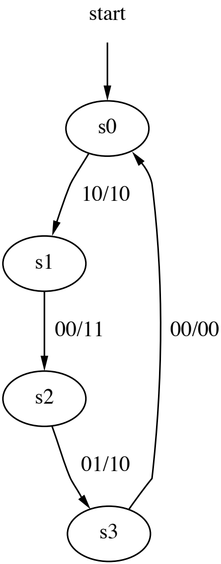
Finite State Machine Diagrams

- FSM's are often represented using a labeled digraph.
- The vertex set contains the states (i.e., $V = S$).
- The edge set contains the set of state transitions (i.e., $(u, v) \in E$ iff $\exists i \in I$ s.t. $((u, i), v) \in \delta$).
- The labeling function is defined by next-state and output functions.
  - Each edge (u, v) is labeled with i/o where $i \in I$ and $o \in O$ and $((u, i), v) \in \delta$ and $((u, i), o) \in \lambda$.

Passive/Active Shop

```
shop_PA_1:process
begin
  guard(req_wine,'1');          - winery calls
  assign(ack_wine,'1',1,3);   - receives wine
  guard(req_wine,'0');          - req_wine reset
  assign(req_patron,'1',1,3); - call patron
  guard(ack_patron,'1');        - wine purchased
  assign(req_patron,'0',1,3); - reset req_patron
  guard(ack_patron,'0');        - ack_patron reset
  assign(ack_wine,'0',1,3);   - reset ack_wine
end process;
```

Passive/Active Shop FSM

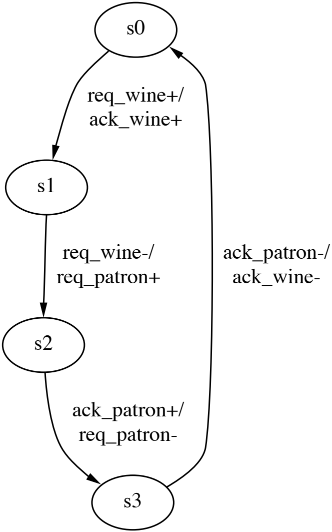


req_wine / ack_patron

	00	01	11	10
s0	s0, 00	–	–	s1, 10
s1	s2, 11	–	–	s1, 10
s2	s2, 11	s3, 10	–	–
s3	s0, 00	s3, 10	–	–

ack_wine / req_patron

Burst-Mode State Machine



req_wine / ack_patron

	00	01	11	10
s0	s0, 00	—	—	s1, 10
s1	s2, 11	—	—	s1, 10
s2	s2, 11	s3, 10	—	—
s3	s0, 00	s3, 10	—	—

ack_wine / req_patron

Burst-Mode State Machines

- V is a finite set of vertices (or states);
- $E \subseteq V \times V$ is the set of edges (or transitions);
- $I = \{x_1, \ldots, x_m\}$ is the set of inputs;
- $O = \{z_1, \ldots, z_n\}$ is the set of outputs;
- $v_0 \in V$ is the start state;
- $in : V \rightarrow \{0, 1\}^m$ is value of the m inputs at entry to state;
- $out : V \rightarrow \{0, 1\}^n$ is value of the n outputs at entry to state.

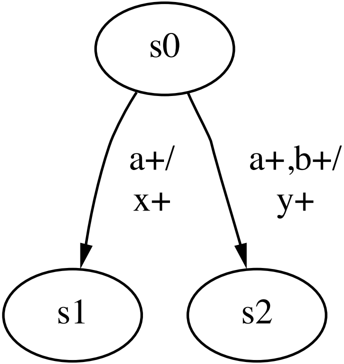
Input and Output Bursts

- *Input burst* is defined by $trans_i : E \longrightarrow 2^I$.
  - $x_i \in trans_i(e)$ *iff* $in_i(u) \neq in_i(v)$
- *Output burst* is defined by $trans_o : E \longrightarrow 2^O$.
  - $x_i \in trans_o(e)$ *iff* $out_i(u) \neq out_i(v)$

Maximal Set Property

- No input burst leaving a given state can be a subset of another leaving the same state.
- The behavior in such a state would be ambiguous.
- $\forall (u, v), (u, w) \in E : trans_i(u, v) \subseteq trans_i(u, w) \Rightarrow v = w$.
- This restrication is called the *maximal set property*.

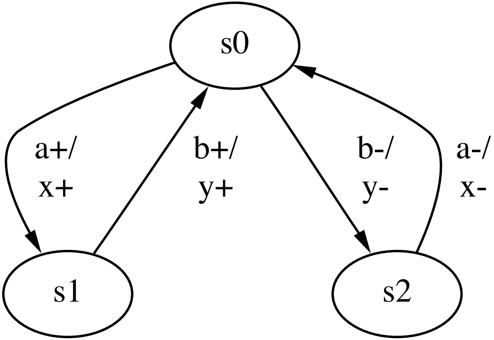
Maximal Set Property

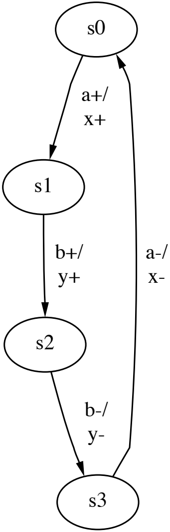


BM State Diagrams

- Not every *BM state diagram* represents a legal BM machine.
- If mislabeled with transitions that are not possible, it is impossible to define the *in* and *out* functions.
- There must be a strict alternation of rising and falling transitions on every input and output signal, across all paths.

BM State Diagrams





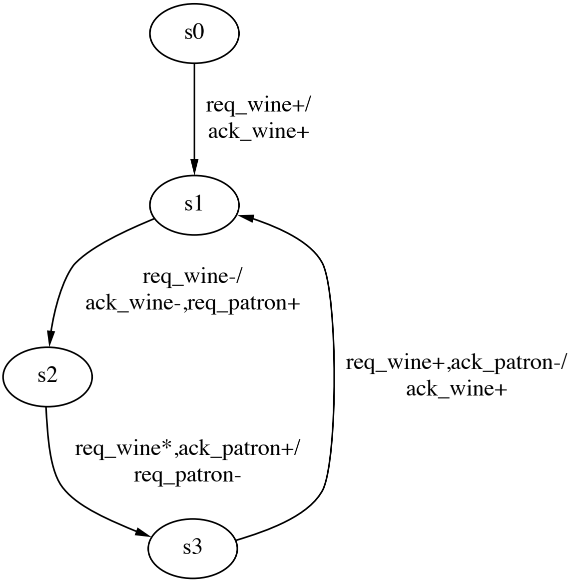
Extended Burst-Mode

- BM machines require prescribed order: inputs change, outputs change, and state signals change.
- In *extended burst-mode (XBM)* state machines, this limitation is loosened a bit by the introduction of *directed don't cares*.
- These allow one to specify that an input change may or may not happen in a given input burst.
- BM machines also are unable to express conditional behavior.
- To support this type of behavior, XBM machines allow *conditional input bursts*.

Directed Don't Cares

```
Shop_PA_2:process
begin
  guard(req_wine,'1');          - winery calls
  assign(ack_wine,'1',1,3);   - receives wine
  guard(req_wine,'0');          - req_wine reset
  assign(ack_wine = '0',1,3,req_patron,'1',1,3);
  guard(ack_patron,'1');        - wine purchased
  assign(req_patron,'0',1,3); - reset req_patron
  guard(ack_patron,'0');        - ack_patron reset
end process;
```

Directed Don't Cares



Directed Don't Cares

req_wine / ack_patron

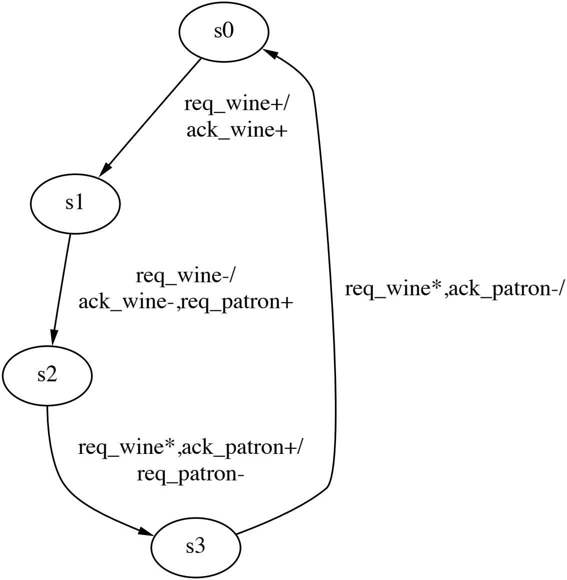
	00	01	11	10
s0	s0, 00	–	–	s1, 10
s1	s2, 01	–	–	s1, 10
s2	s2, 01	s3, 00	s3, 00	s2, 01
s3	s3, 00	s3, 00	s3, 00	s1, 10

ack_wine / req_patron

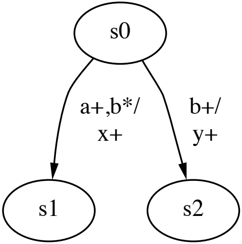
Directed Don't Cares

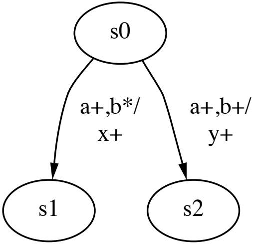
- A transition is *terminating* when it is of the form $t+$ or $t-$.
- A directed don't care transition is of the form $t*$.
- A *compulsory* transition is a terminating transition which is not preceded by a directed don't care transition.
- Each input burst must have at least one compulsory transition.

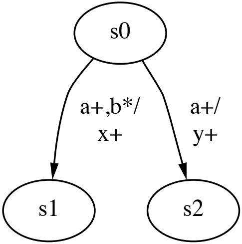
Directed Don't Cares



Modified Maximal Set Property







Conditional Input Bursts

```
Shop_PA_2:process
begin
  guard(req_wine,'1');
  shelf <= bottle after delay(2,4);
  wait for delay(5,10);
  assign(ack_wine,'1',1,3);
  guard(req_wine,'0');
  if (shelf = '0') then
    assign(ack_wine,'0',1,3,req_patron1,'1',1,3);
    guard(ack_patron1,'1');
    assign(req_patron1,'0',1,3);
    guard(ack_patron1,'0');
```

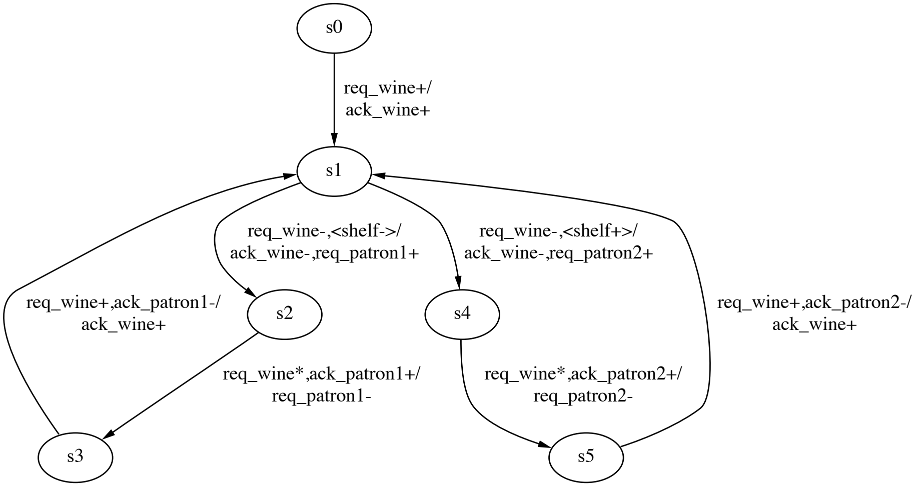
Conditional Input Bursts

```
  elsif (shelf = '1') then
    assign(ack_wine,'0',1,3,req_patron2,'1',1,3);
    guard(ack_patron2,'1');
    assign(req_patron2,'0',1,3);
    guard(ack_patron2,'0');
  end if;
end process;
```

Conditional Input Bursts

- A conditional input burst includes a regular input burst and a *conditional clause*.
- A clause of the form $< s- >$ indicates that the transition is only taken if s is low.
- A clause of the form $< s+ >$ indicates that the transition is only taken if s is high.
- The signal in the conditional clause must be stable before every compulsory transition in the input burst.

Conditional Input Bursts



Conditional Input Bursts

req_wine / ack_patron1 / ack_patron2 / shelf

	0000	0001	0011	0010	0110	0111	0101	0100
s0	s0, 000	s0, 000	–	–	–	–	–	–
s1	s2, 010	s4, 001	–	–	–	–	–	–
s2	s2, 010	s2, 010	–	–	–	–	s3, 000	s3, 000
s3	s3, 010	s3, 010	–	–	–	–	s3, 000	s3, 000
s4	s4, 001	s4, 001	s5, 000	s5, 000	–	–	–	–
s5	s5, 000	s5, 000	s5, 000	s5, 000	–	–	–	–

ack_wine / req_patron1 / req_patron2

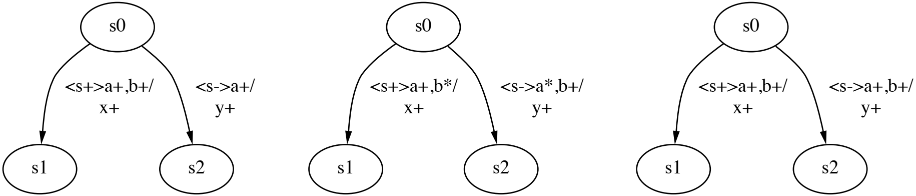
Conditional Input Bursts

req_wine / ack_patron1 / ack_patron2 / shelf

	1100	1101	1111	1110	1010	1011	1001	1000
s0	–	–	–	–	–	–	s1, 100	s1, 100
s1	–	–	–	–	–	–	s1, 100	s1, 100
s2	s3, 000	s3,000	–	–	–	–	s2, 010	s2, 010
s3	s3, 000	s3, 000	–	–	–	–	s1, 100	s1, 100
s4	–	–	–	–	s5, 000	s5,000	s4, 001	s4, 001
s5	–	–	–	–	s5, 000	s5, 000	s1, 100	s1, 100

ack_wine / req_patron1 / req_patron2

Modified Maximal Set Property



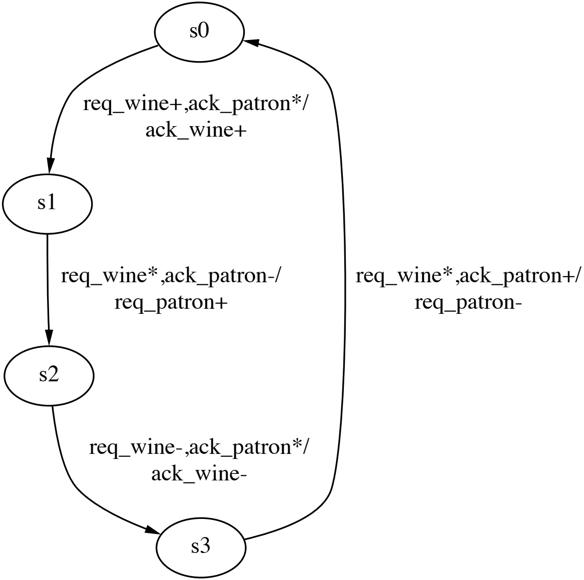
Burst-Mode State Machines

- V is a finite set of vertices (or states).
- $E \subseteq V \times V$ is the set of edges (or transitions).
- $I = \{x_1, \ldots, x_m\}$ is the set of inputs.
- $O = \{z_1, \ldots, z_n\}$ is the set of outputs.
- $C = \{c_1, \ldots, c_l\}$ is the set of conditional signals.
- $v_0 \in V$ is the start state.
- $in : V \rightarrow \{0, 1, *\}^m$ defines m inputs upon entry to each state.
- $out : V \rightarrow \{0, 1\}^n$ defines n outputs upon entry to each state.
- $cond : E \rightarrow \{0, 1, *\}^l$ defines needed conditional inputs.

No XBM Machine

```
Shop_PA_lazy_active:process
begin
  guard(req_wine,'1');        - winery calls
  assign(ack_wine,'1',1,3);  - receives wine
  guard(ack_patron,'0');      - ack_patron reset
  assign(req_patron,'1',1,3); - call patron
  guard(req_wine,'0');        - req_wine reset
  assign(ack_wine,'0',1,3);  - reset ack_wine
  guard(ack_patron,'1');      - wine purchased
  assign(req_patron,'0',1,3); - reset req_patron
end process;
```

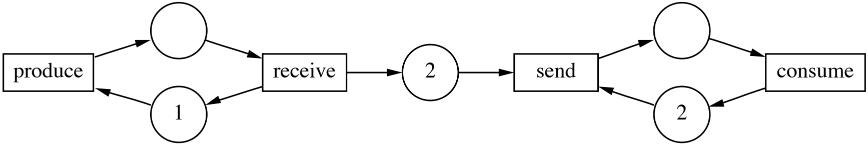
Illegal XBM Machine



Petri-Nets

- A *Petri-net* is a bipartite digraph.
- The vertex set is partitioned into two disjoint subsets:
  - P is the set of *places*.
  - T is the set of *transitions*.
- The set of arcs, F, is composed of pairs where one element is from P and the other is from T (i.e., $F \subseteq (P \times T) \cup (T \times P)$).
- A Petri-net is $\langle P, T, F, M_0 \rangle$ where M_0 is the initial marking.

Petri-net for Shop with Infinite Shelf Space



Presets and Postsets

- The *preset* of a transition $t \in T$ (denoted $\bullet t$) is the set of places connected to t (i.e., $\bullet t = \{p \in P \mid (p, t) \in F\}$).
- The *postset* of a transition $t \in T$ (denoted $t\bullet$) is the set of places t is connected to (i.e., $t\bullet = \{p \in P \mid (t, p) \in F\}$).
- The preset of a place $p \in P$ (denoted $\bullet p$) is the set of transitions connected to p (i.e., $\bullet p = \{t \in T \mid (t, p) \in F\}$).
- The postset of a place $p \in P$ (denoted $p\bullet$) is the set of transitions p is connected to (i.e., $p\bullet = \{t \in T \mid (p, t) \in F\}$).

Markings

- A *marking*, M, for a Petri net is a function that maps places to natural numbers (i.e., $M : P \rightarrow N$).
- Markings can be added or subtracted using vector arithmetic.
- They can also be compared:

$$M \geq M' \quad \textit{iff} \quad \forall p \in P \,.\, M(p) \geq M'(p)$$

- For a set of places, $A \subseteq P$, C_A denotes the *characteristic marking* of A:

$$C_A(p) \quad = \quad \textit{if } p \in A \textit{ then } 1 \textit{ else } 0.$$

Transition Firings

- A transition t is *enabled* under the marking M if $M \geq C_{\bullet t}$.
- In other words, $M(p) \geq 1$ for each $p \in \bullet t$.
- The firing transforms the marking as follows (denoted $M[t\rangle M'$):

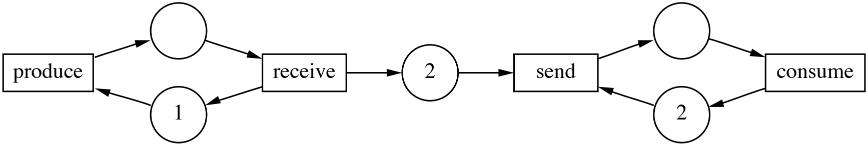
$$M' = M - C_{\bullet t} + C_{t \bullet}$$

- When a transition t fires, a token is removed from each place in its preset, and a token is added to each place in its postset.

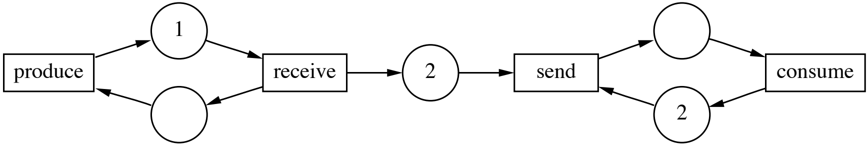
Reachable Markings

- Firing of a transition transforms the marking of the Petri net into a new marking.
- A sequence of transition firings ($\sigma = t_1, t_2, \ldots, t_n$) produces a sequence of markings (M_0, M_1, ..., M_n).
- If such a *firing sequence* exists, we say that the marking M_n is *reachable* from M_0 by σ (denoted $M_0[\sigma\rangle M_n$).
- We denote the set of all markings reachable from a given marking by $[M\rangle$.

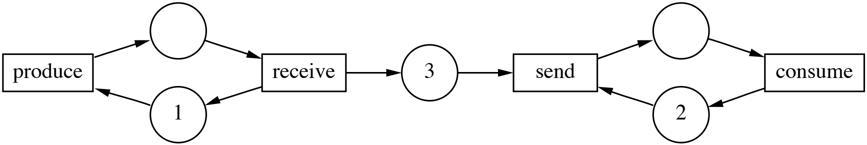
Example Firing Sequence



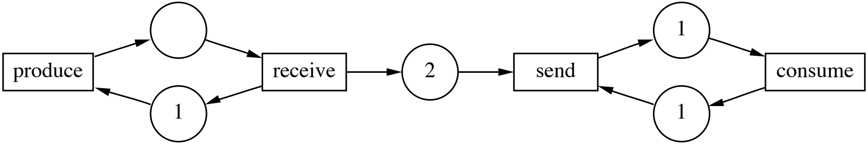
Example Firing Sequence



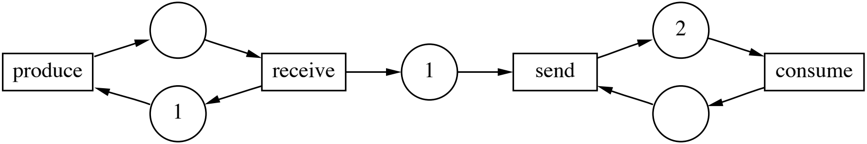
Example Firing Sequence



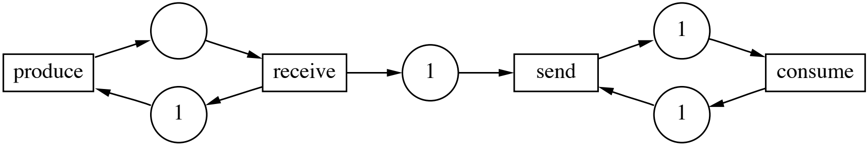
Example Firing Sequence



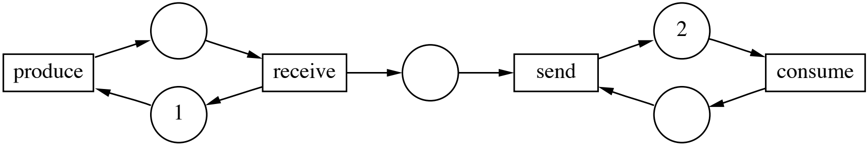
Example Firing Sequence



Example Firing Sequence



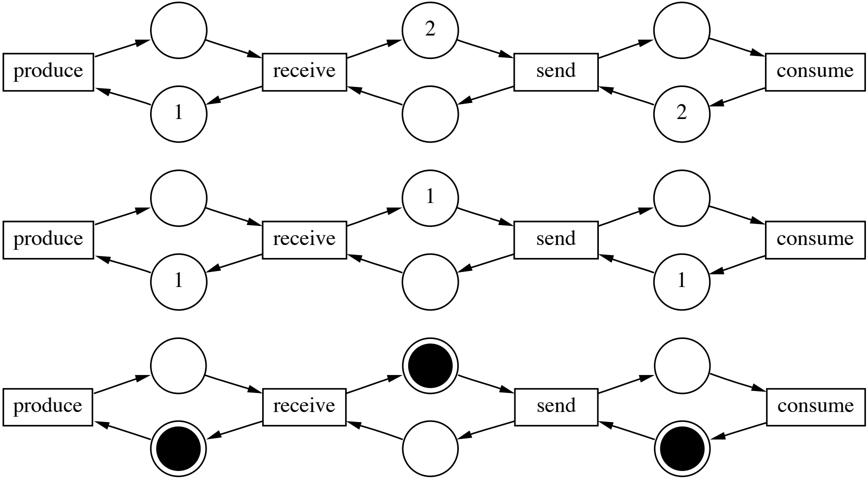
Example Firing Sequence



k-Bounded Petri-Nets

- A Petri net is *k-bounded* if there does not exist a reachable marking which has a place with more than k tokens.
- A 1-bounded Petri net is also called a *safe* Petri net (i.e., $\forall p \in P, \forall M \in [M_0\rangle . M(p) \leq 1$).
- When working with safe Petri nets, a marking can be denoted as simply a subset of places.
- If $M(p) = 1$, $p \in M$, and if $M(p) = 0$, we $p \notin M$.
- $M(p)$ cannot take on any other values in a safe Petri net.
- Since a marking can only take on the values 1 and 0, the place can be annotated with a token when 1 and without when 0.

k-Bounded Petri-net



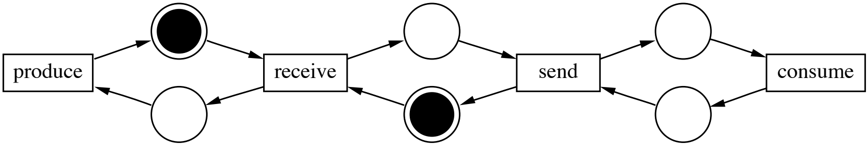
Liveness

- A Petri net is *live* if from every reachable marking, there exists a sequence of transitions such that any transition can fire.

$$\forall M \in [M_0\rangle, \forall t \in T, \exists M' \in [M\rangle . M' \geq C_{\bullet t}$$

- To determine if a Petri net is live, it is typically necessary to find all the reachable markings.

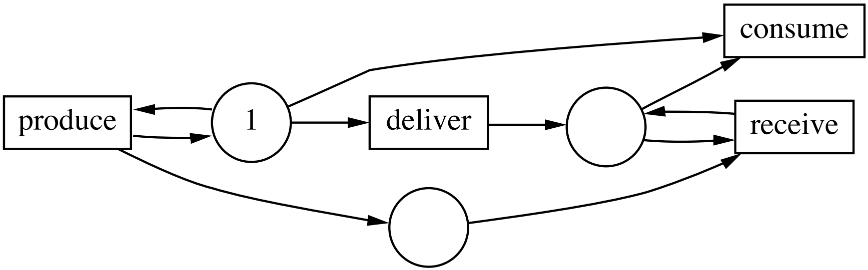
Liveness



Liveness Categories

- Different liveness categories can be determined more easily.
- In particular, a transition t for a given Petri net is said to be:
  1. *dead (L0-live)* if there does not exist a firing sequence in which t can be fired.
  2. *L1-live (potentially firable)* if there exists at least one firing sequence in which t can be fired.
  3. *L2-live* if t can be fired at least k times.
  4. *L3-live* if t can be fired infinitely often in some firing sequence.
  5. *L4-live* or *live* if t is L1-live in every marking reachable from the initial marking.
- A Petri net is *Lk-live* if every transition in the net is Lk-live.

Liveness Categories



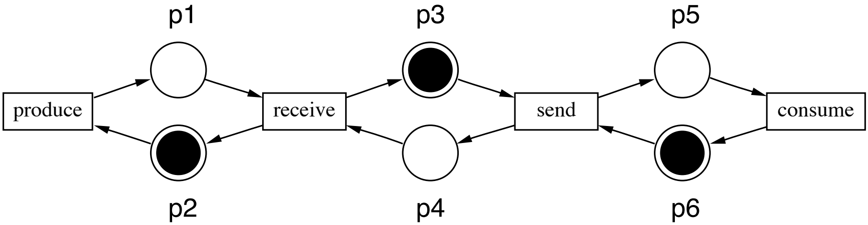
Reachability Graph

- When a Petri net is bounded, the number of reachable markings is finite, and a *reachability graph* (RG) can be found.
- In an RG, the vertices, Φ, are the markings and the edges, Γ, are the possible transition firings between two markings.
- For safe Petri nets, vertices in RG are labeled with the subset of places included in the marking.
- The edges are labeled with the transition that fires to move the Petri net from one marking to the next.

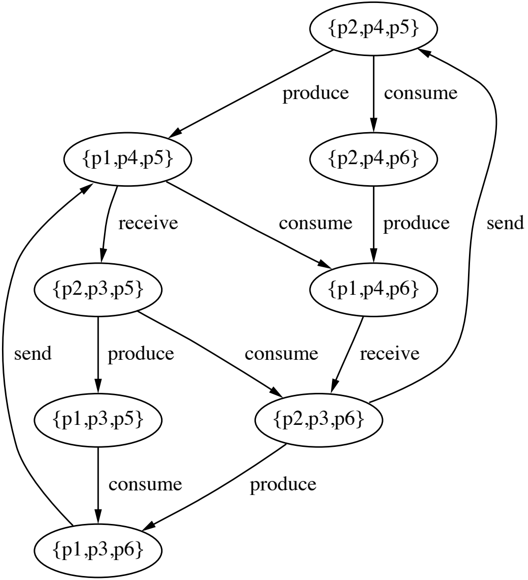
Algorithm to Find Reachability Graph

```
find_RG(Petri net ⟨P, T, F, M_0⟩)
  M = M_0;  T_e = {t ∈ T | M ≥ C_•t};  Φ = {M};  Γ = ∅;
  done = false;
  while (¬ done)
    t = select(T_e);
    if (T_e − {t} ≠ ∅) then push(M, T_e − {t});
    M′ = M − C_•t + C_t•;
    if (M′ ∉ Φ) then
      Φ = Φ ∪ {M′};  Γ = Γ ∪ {(M, M′)};
      M = M′;  T_e = {t ∈ T | M ≥ C_•t};
    else
      Γ = Γ ∪ {(M, M′)};
      if (stack is not empty) then (M, T_e) = pop();
      else done = true;
  return(Φ, Γ);
```

Safe Example



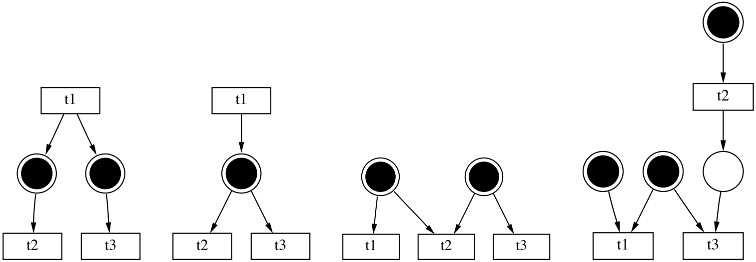
Example Reachability Graph



Concurrency, Conflict, and Confusion

- Two transitions t_1 and t_2 are *concurrent* when there exists markings where both are enabled and can fire in either order.
- Two transitions, t_1 and t_2, are in *conflict* when the firing of one disables the firing of the other.
- When concurrency and conflict are mixed, we get *confusion*.

Example of Concurrency, Conflict, and Confusion



State Machines and Marked Graphs

- A Petri-net is a *state machine* if and only if every transition has exactly one place in its preset and one place in its postset.

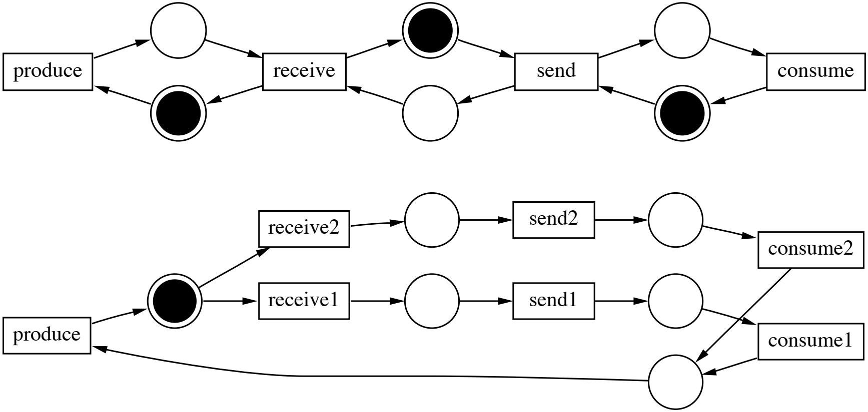
$$\forall t \in T : |\bullet t| = |t \bullet| = 1$$

- State machines do not allow concurrency, but do allow conflict.
- A Petri-net is a *marked graph* if and only if every place has exactly one transition in its preset and one in its postset.

$$\forall p \in P : |\bullet p| = |p \bullet| = 1$$

- Marked graphs do not allow conflict, but do allow concurrency.

Example Nets



Free Choice Nets

- A Petri-net is *free choice* if and only if every pair of transitions that share a common place in their preset have only a single place in their preset.

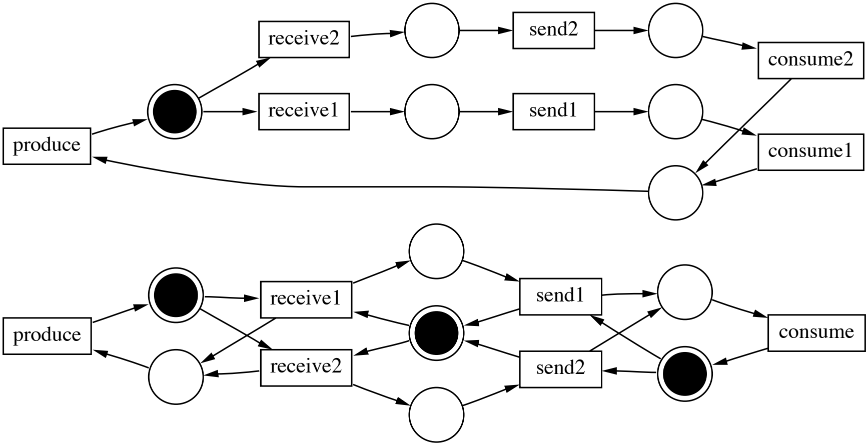
$$\forall t, t' \in T, t \neq t' : \bullet t \cap \bullet t' \neq \emptyset \Rightarrow |\bullet t| = |\bullet t'| = 1$$

$$\forall p, p' \in P, p \neq p' : p\bullet \cap p'\bullet \neq \emptyset \Rightarrow |p\bullet| = |p'\bullet| = 1$$

$$\forall p \in P, \forall t \in T : (p, t) \in F \Rightarrow p\bullet = \{t\} \vee \bullet t = \{p\}$$

- Free choice nets allow concurrency and conflict, but do allow confusion.

Example Nets



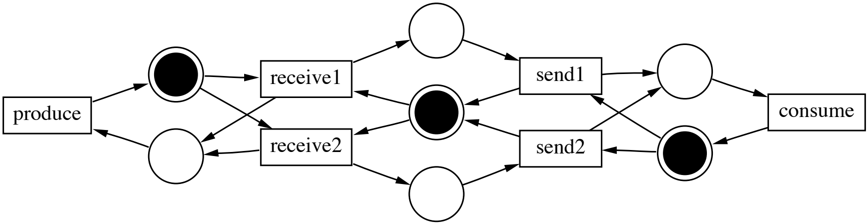
Extended Free Choice Nets

- A Petri net is an *extended free choice net* if and only if every pair of places that share common transitions in their postset have exactly the same transitions in their postset.

$$\forall p, p' \in P \; . \; p\bullet \cap p'\bullet \neq \emptyset \Rightarrow p\bullet = p'\bullet$$

- Extended free-choice nets also allow concurrency and conflict, but they do not allow confusion.

Example Nets



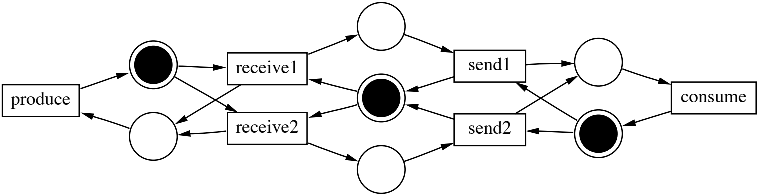
Asymmetric Choice Nets

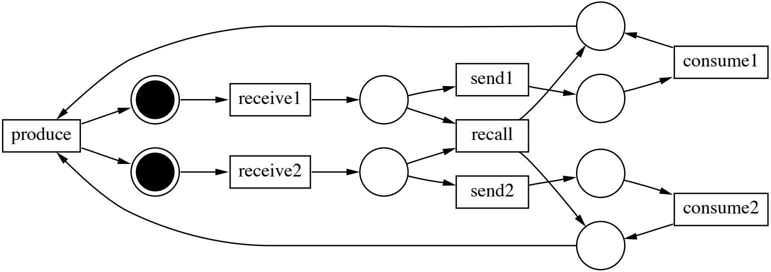
- A Petri net is an *asymmetric choice net* if and only if for every pair of places that share common transitions in their postset, one has a subset of the transitions of the other.

$$\forall p, p' \in P \;.\; p\bullet \cap p'\bullet \neq \emptyset \Rightarrow p\bullet \subseteq p'\bullet \vee p'\bullet \subseteq p\bullet$$

- Asymmetric choice nets allow *asymmetric confusion* but not *symmetric confusion*.

Example Nets





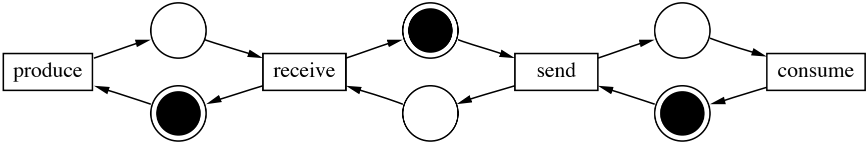
Checking Safety and Liveness

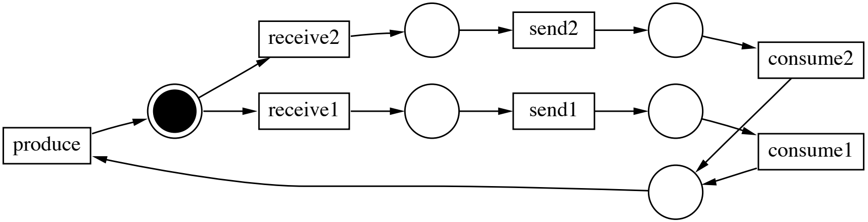
- It is possible to check safety and liveness for certain restricted classes of Petri nets using the theorems given below.

  Theorem 4.1 A state machine is live and safe iff it is strongly connected and M_0 has exactly one token.

  Theorem 4.2 (Commoner, 1971) A marked graph is live and safe iff it is strongly connected and M_0 places exactly one token on each simple cycle.

Example Nets

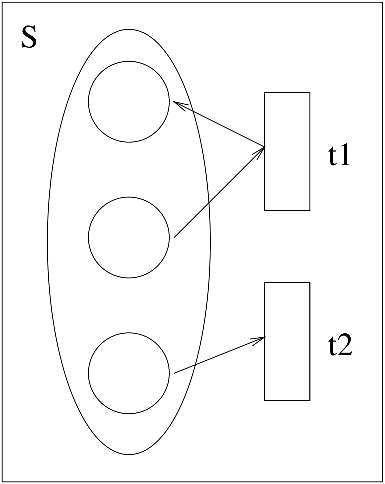


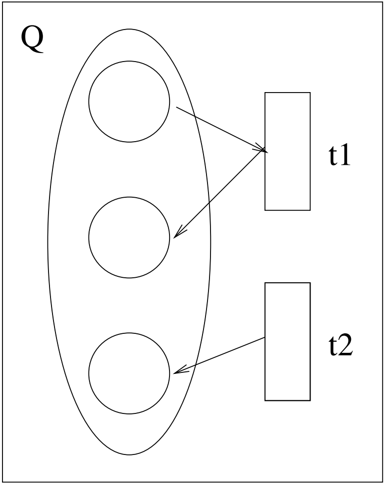


Siphons and Traps

- A *siphon* is a nonempty subset of places, S, in which every transition having a postset place in S also has a preset place in S (i.e., $\bullet S \subseteq S\bullet$).
- If in some marking no place in S has a token, then in all future markings, no place in S will ever have a token.
- A *trap* is a nonempty subset of places, Q, in which every transition having a preset place in Q also has a postset place in Q (i.e., $Q\bullet \subseteq \bullet Q$).
- If in some marking some place in Q has a token, then in all future markings some place in Q will have a token.

Example Siphon and Trap





Checking Liveness

Theorem 4.3 (Hack, 1972) A free-choice net, N, is live iff every siphon in N contains a marked trap.

Theorem 4.4 (Commoner, 1972) An asymmetric choice net N is live if (but not only if) every siphon in N contains a marked trap.

State Machine Components

- A *state machine component* of a net, N, is a subnet in which each transition has at most one place in its preset and one place in its postset and is generated by these places.
- The net generated by a set of places includes these places, all transitions in their preset and postset, and all connecting arcs.
- A net N is said to be covered by a set of SM-components when the set of components includes all places, transitions, and arcs from N.

  Theorem 4.5 (Hack, 1972) A live free-choice net, N, is safe iff N is covered by strongly connected SM-components each of which has exactly one token in M_0.

Marked Graph Components

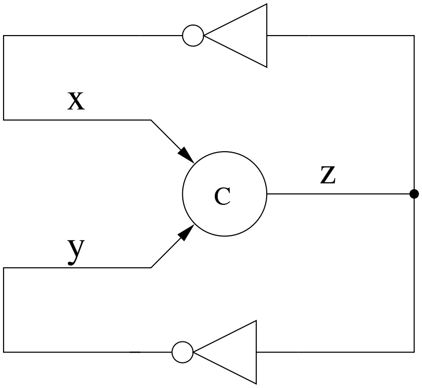
- A *marked graph component* of a net, N, is a subnet in which each place has at most one transition in its preset and one transition in its postset and is generated by these transitions.
- The net generated by a set of transitions includes these transitions, all places in their preset and postset, and all connecting arcs.
- A net N is said to be covered by a set of MG-components when the set of components includes all places, transitions, and arcs from N.

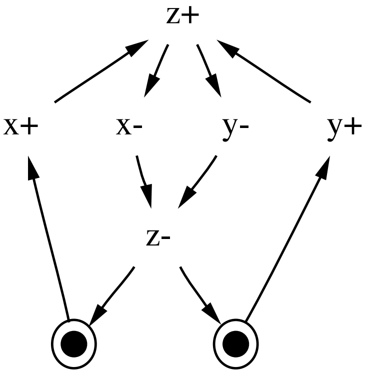
  Theorem 4.6 If N is a live and safe free-choice net then N is covered by strongly connected MG-components.

Signal Transition Graphs (STG)

- To use a Petri net to model asynchronous circuits, must relate transitions to events on signal wires.
- Several variants of Petri nets accomplish this: *M-nets*, *I-nets*, and *change diagrams*.
- A *signal transition graph* (STG) is a labeled safe Petri net which is modeled by $\langle P, T, F, M_0, N, s_0, \lambda_T \rangle$, where:
  - $N = I \cup O$ is the set of signals where I is the set of input signals and O is the set of output signals.
  - s_0 is the initial value for each signal in the initial state.
  - $\lambda_T : T \rightarrow N \times \{+, -\}$ is the *transition labeling function*.
- Each transition is labeled with either a rising transition, $s+$, or falling transition, $s-$.
- A STG imposes explicit restrictions on the environment.

Example Signal Transition Graph (STG)

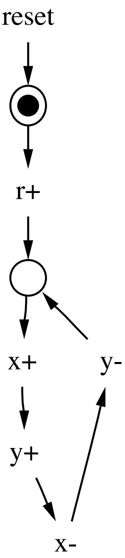




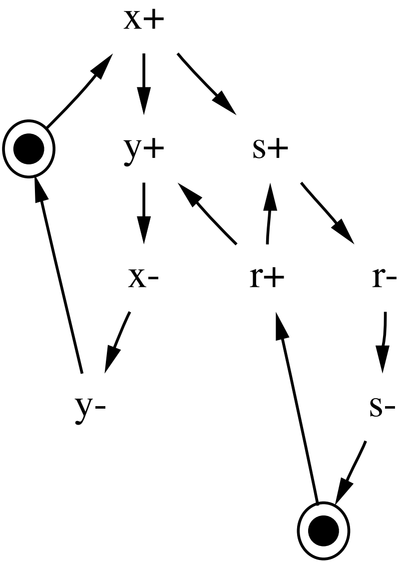
STG Restrictions

- STGs are often restricted to a synthesizable subset.
- Synthesis methods often restrict the STG to be *live* and *safe*.
- Some synthesis methods require STGs to be *persistent*.
- A STG is persistent if for all $a* \rightarrow b*$, there exist other arcs that ensure that $b*$ fires before the opposite transition of $a*$.
- Other methods require *single-cycle transitions*.
- A STG has single-cycle transitions if each signal name appears in exactly one rising and one falling transition.
- None of these restrictions is actually a necessary requirement for a circuit implementation to exist.
- These restrictions can simplify the synthesis algorithms.

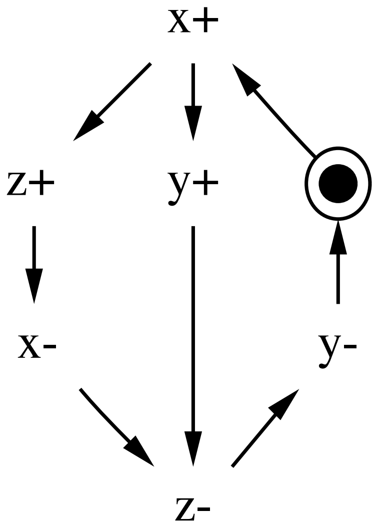
Liveness



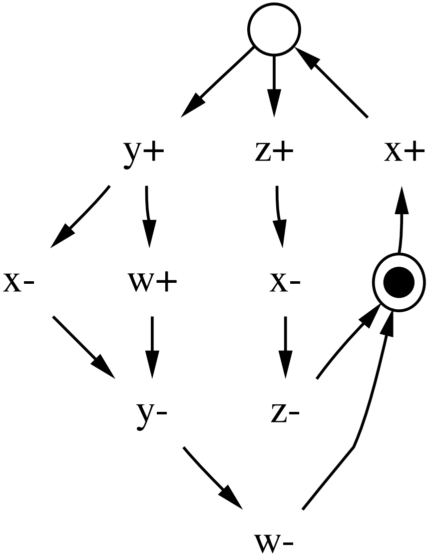
Safety



Persistency



Single-Cycle Transitions



State Graphs (SG)

- To design a circuit from an STG, must find its *state graph*.
- A SG is modeled by the tuple $\langle\ S, \delta, \lambda_S\ \rangle$.
  - S is the set of states.
  - $\delta \subseteq S \times T \times S$ is the set of state transitions.
  - $\lambda_S : S \rightarrow (N \rightarrow \{0, 1\})$ is the *state labeling function*.
- Each state s is labeled with a vector $\langle s(0), s(1), \ldots, s(n)\rangle$, where $s(i)$ is either 0 or 1, indicating value returned by λ_S.
- We use $s(i)$ interchangeably with $\lambda_S(s)(i)$.

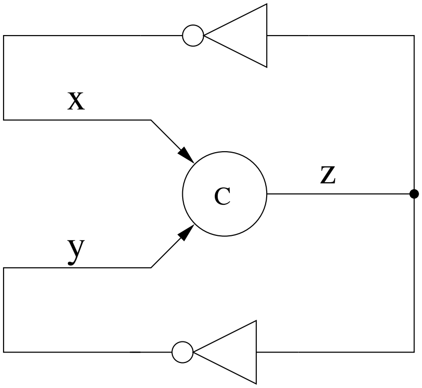
Implied State

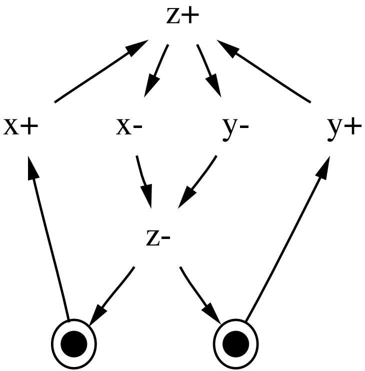
- If in s_i, there exists a transition on signal u_i to s_j [i.e., $\exists(s_i, t, s_j) \in \delta \,.\, \lambda_T(t) = u_i+ \vee \lambda_T(t) = u_i-$], then u_i is *excited*.
- Otherwise, the signal u_i is in *equilibrium*.
- The value each signal is tending to is called its *implied value*.
- If the signal is excited, the implied value of u_i is $\overline{s(i)}$.
- If the signal is in equilibrium, the implied value of u_i is $s(i)$.
- The *implied state*, s' is labeled with a binary vector $\langle s'(0), s'(1), \ldots, s'(n)\rangle$ of the implied values.
- The function $X : S \rightarrow 2^N$ returns the set of excited signals in a given state [i.e., $X(s) = \{u_i \in S \mid s(i) \neq s'(i)\}$].
- When $u_i \in X(s)$ and $s(i) = 0$, $s(i)$ in SG is marked with “R”.
- When $u_i \in X(s)$ and $s(i) = 1$, $s(i)$ in SG is marked with “F”.

Algorithm to Find SG

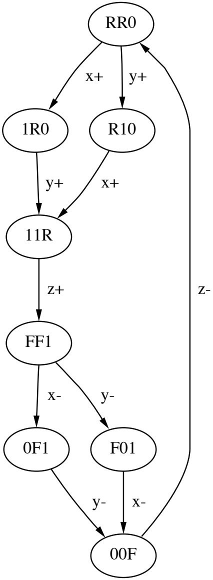
find_SG($\langle P, T, F, M_0, N, s_0, \lambda_T \rangle$)
  $M = M_0$; $s = s_0$; $S = \{M\}$; $\lambda_S(M) = s$;
  $T_e = \{t \in T | M \subseteq \bullet t\}$; `done = false`;
  while ($\neg$ `done`)
    $t =$ `select`(T_e);
    if ($T_e - \{t\} \neq \emptyset$) **then** **push**($M, s, T_e - \{t\}$);
    if ($(M - \bullet t) \cap t\bullet \neq \emptyset$) **then** **return**("`Not safe.`");
    $M' = (M - \bullet t) \cup t\bullet$; $s' = s$;
    if ($\lambda_T(t) = u+$) **then** $s'(u) = 1$;
    else if ($\lambda_T(t) = u-$) **then** $s'(u) = 0$;
    if ($M' \notin S$) **then**
      $S = S \cup \{M'\}$; $\lambda_S(M') = s'$ $\delta = \delta \cup \{(M, t, M')\}$;
      $M = M'$; $s = s'$; $T_e = \{t \in T | M \subseteq \bullet t\}$;
    else
      if ($\lambda_S(M') \neq s'$) **then** **return**("`Inconsistent.`");
      if (`stack is not empty`) **then** (M, s, T_e) = **pop**();
      else `done = true`;
  return($\langle S, \delta, \lambda_S \rangle$);

Example Signal Transition Graph (STG)





SG for C-Element



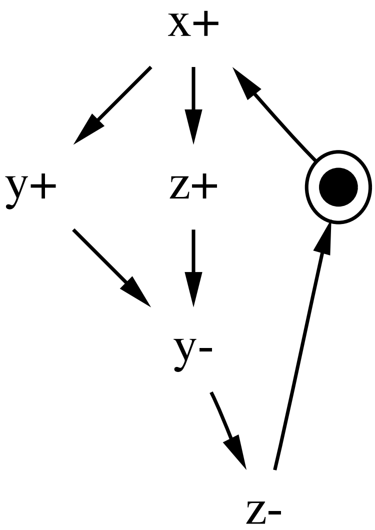
Consistent State Assignment

- A well-formed SG, must have a *consistent state assignment*.
- A SG has a consistent state assignment if for each state transition $(s_i, t, s_j) \in \delta$ exactly one signal changes value, and its value is consistent with the transition.

$$
\begin{aligned}
\forall (s_i, t, s_j) \in \delta . \forall u \in N \quad & . \quad (\lambda_T(t) \neq u* \wedge s_i(u) = s_j(u)) \\
& \vee \quad (\lambda_T(t) = u+ \wedge s_i(u) = 0 \wedge s_j(u) = 1) \\
& \vee \quad (\lambda_T(t) = u- \wedge s_i(u) = 1 \wedge s_j(u) = 0)
\end{aligned}
$$

  where "$*$" represents either "$+$" or "$-$".
- A STG produces a SG with a consistent state assignment if in any firing sequence, transitions of a signal strictly alternate between $+$'s and $-$'s.

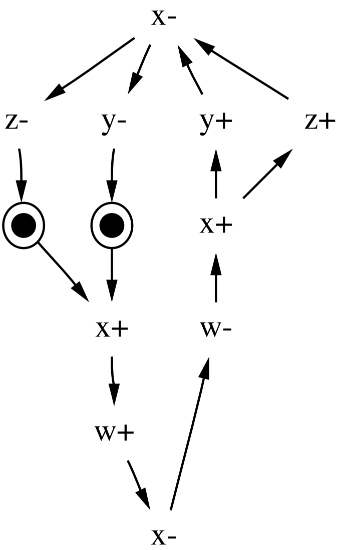
Consistent State Assignment



Unique State Code

- A SG has a *unique state assignment* (USC) if no two different states (i.e., markings) have identical values for all signals [i.e., $\forall s_i, s_j \in S, s_i \neq s_j \, . \, \lambda(s_i) \neq \lambda(s_j)$].
- Some synthesis methods are restricted to STGs that produce SGs with USC.

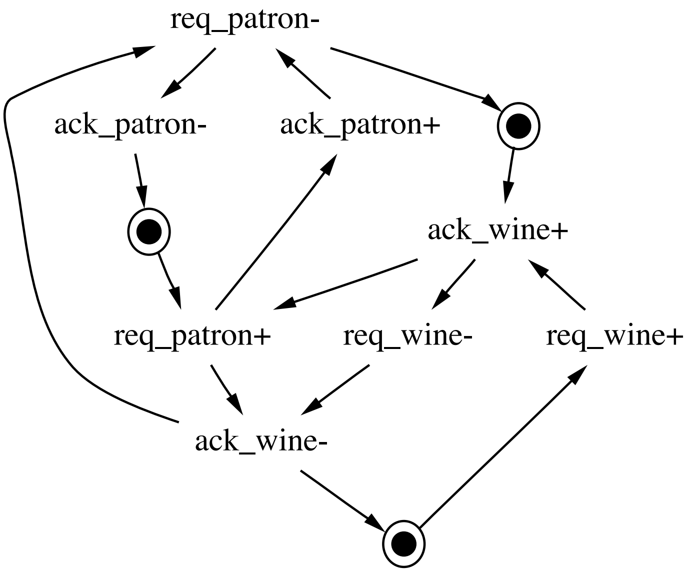
Unique State Code



Reshuffled Passive/Lazy-Active Wine Shop

```
Shop_PA_lazy_active:process
begin
  guard(req_wine,'1');         - winery calls
  assign(ack_wine,'1',1,3);  - receives wine
  guard(ack_patron,'0');       - ack_patron reset
  assign(req_patron,'1',1,3); - call patron
  guard(req_wine,'0');         - req_wine reset
  assign(ack_wine,'0',1,3);  - reset ack_wine
  guard(ack_patron,'1');       - wine purchased
  assign(req_patron,'0',1,3); - reset req_patron
end process;
```

STG for Reshuffled Passive/Lazy-Active Wine Shop



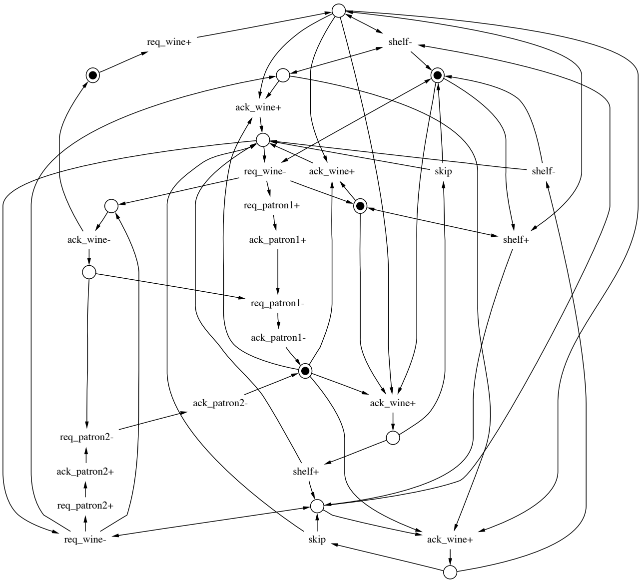
Wine Shop with Two Patrons

```
Shop_PA_2:process
begin
  guard(req_wine,'1');
  shelf <= bottle after delay(2,4);
  wait for delay(5,10);
  assign(ack_wine,'1',1,3);
  guard(req_wine,'0');
  if (shelf = '0') then
    assign(ack_wine,'0',1,3,req_patron1,'1',1,3);
    guard(ack_patron1,'1');
    assign(req_patron1,'0',1,3);
    guard(ack_patron1,'0');
```

Wine Shop with Two Patrons

```
  elsif (shelf = '1') then
    assign(ack_wine,'0',1,3,req_patron2,'1',1,3);
    guard(ack_patron2,'1');
    assign(req_patron2,'0',1,3);
    guard(ack_patron2,'0');
  end if;
end process;
```

STG for Wine Shop with Two Patrons



Labeled Petri nets

- AFSMs cannot model arbitrary concurrency.
- Petri-nets have difficulty to express signal levels.
- *Labeled Petri nets* are a hybrid graphical representation method which are both capable of modelling arbitrary concurrency and signal levels.

Labeled Petri Nets (LPN)

- A LPN is a tuple $\langle P, T, B, F, L, M_0, S_0 \rangle$ where:
  - P is a finite set of places;
  - T is a finite set of transitions;
  - B is a finite set of Boolean variables;
  - $F \subseteq (P \times T) \cup (T \times P)$ is the flow relation;
  - L is a tuple of labels;
  - $M_0 \subseteq P$ is the set of initially marked places; and
  - S_0 is the set of initial Boolean variable values.

Labels

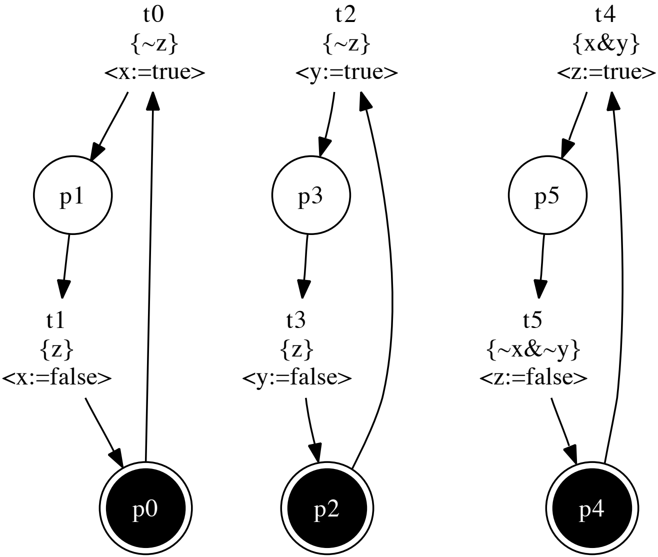
- Each transition $t \in T$ has the following labels:
  - $En : T \rightarrow \mathcal{P}$ - the enabling condition;
  - $D : T \rightarrow \mathbb{Q} \times (\mathbb{Q} \cup \{\infty\})$ - the delay asssignment; and
  - $BA : T \times B \rightarrow \mathcal{P}$ - Boolean variable assignments.
- The language for the $\mathcal{P}$ is defined as follows:

$$\phi \quad ::= \quad \textbf{true} \mid \textbf{false} \mid b_i \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

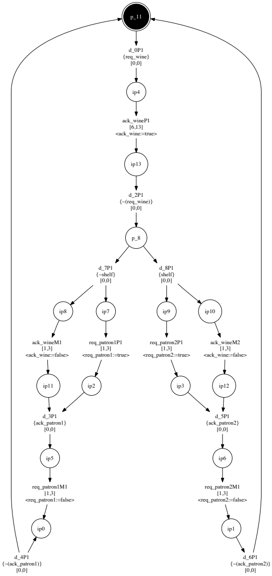
Semantics

- A transition t is enabled to fire when its preset ($\bullet t$) is marked and its enabling condition ($En(t)$) evaluates to true in the current state.
- Once a transition is enabled it fires sometime between the lower and upper bound associated with its delay assignment ($D(t)$).
- When a transition fires, the marking is updated and the Boolean assignments associated with the transition ($BA(t, v)$) are executed.

LPN for a C-Element



LPN for Wine Shop with Two Patrons



Summary

- Finite state machines (AFSMs, BM, and XBM).
- Petri-nets, STGs, and LPNs.